Def. Open subscheme of X: pair $(U, \mathcal{O}_X|_U)$ where $U \subseteq X$ open.

Def. $j: Y \longrightarrow X$ is an open immersion if

1) the underlying map of top spaces is a homeo onto some open $U \subseteq X$
2) $j^\#: \mathcal{O}_X \longrightarrow j_* \mathcal{O}_Y$ induces an iso $\mathcal{O}_X|_U \longrightarrow \tilde{j}_* \mathcal{O}_Y$ where $\tilde{j}: Y \longrightarrow U$ is the morphism with restricted codomain.

That is, $\tilde{j}: (Y, \mathcal{O}_Y) \longrightarrow (U, \mathcal{O}_X|_U)$ is an iso.

Def. Closed subscheme of X: triple $(Y, \mathcal{O}_Y, i^\#)$ where

- $Y \subseteq X$ is a closed subset, $i: Y \hookrightarrow X$
- $\mathcal{O}_Y$ is a sheaf of rings on X s.t. $(Y, \mathcal{O}_Y)$ is a scheme
- $i^\#: \mathcal{O}_X \to i_* \mathcal{O}_Y$ is surjective ("locally every fn on Y is a restriction of a fn on X")

Def. $i: Z \longrightarrow X$ is a closed immersion if

1) the underlying map of top spaces is a homeo onto a closed subset of X
2) $i^\#: \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Z$ is surjective

Rmk. If $i: Z \to X$ is a closed immersion, $Y := i(Z)$, $\tilde{i}: Z \longrightarrow Y$, $\mathcal{O}_Y := \tilde{i}_* \mathcal{O}_Z$ then $\tilde{i}: Z \xrightarrow{\sim} Y \subseteq X$, Y is a closed subscheme.

Key example. $X = \operatorname{Spec} A$, $\mathfrak{a} \subseteq A$ ideal, $\varphi: A \twoheadrightarrow A/\mathfrak{a}$ quotient map

Then $f_\varphi: \operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ is a closed immersion.

1)
$$\begin{aligned} \operatorname{Spec} A/\mathfrak{a} &\xrightarrow{\cong} V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p}\} \quad \text{homeo} \\ \mathfrak{q} &\longmapsto \varphi^{-1}(\mathfrak{q}) \supseteq \mathfrak{a} \end{aligned}$$

2) Let $\mathfrak{p} \in \operatorname{Spec} A$.

$$A_\mathfrak{p} = \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \twoheadrightarrow f_* \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}, \mathfrak{p}} = \begin{cases} 0 & \text{if } \mathfrak{p} \notin V(\mathfrak{a}) \\ (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}} & \text{if } \mathfrak{p} \in V(\mathfrak{a}) \quad (\Leftrightarrow \mathfrak{a} \subseteq \mathfrak{p}) \end{cases}$$

Surjectivity on stalks $\Rightarrow \mathcal{O}_{\operatorname{Spec} A} \twoheadrightarrow f_* \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}}$

Thm. $X := \operatorname{Spec} R$, $Z \overset{f}{\hookrightarrow} X$ closed subscheme. $I := \operatorname{Ker}\left(R \xrightarrow{f^\#\ \operatorname{Spec} R} \mathcal{O}_Z(Z)\right)$.

Then
$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\circlearrowleft"] & \operatorname{Spec} R \\ & \operatorname{Spec} R/I \arrow[ur, hook, "j"'] & \end{tikzcd}$$

In particular, every closed subscheme of an affine scheme is affine.

Pf: Step 1.

$$\begin{tikzcd} R \arrow[rr, "f^\#(\operatorname{Spec} R) = \varphi"] \arrow[dr] & \circlearrowleft & \mathcal{O}_Z(Z) \\ & R/I \arrow[ur, "\psi"'] & \end{tikzcd}$$

f factors as $Z \xrightarrow{\tilde{f}} \operatorname{Spec} R/I \xrightarrow{j} \operatorname{Spec} R$ since $\operatorname{Hom}(X, \operatorname{Spec} R) = \operatorname{Hom}(R, \mathcal{O}_X(X))$.

Claim. $\tilde{f}$ is a closed immersion.

1) $|\tilde{f}|$ is a homeo onto its img

$f(Z) \subseteq \operatorname{Spec} R$ closed $\Rightarrow$ $f(Z) \subseteq \operatorname{Spec}(R/I)$ closed

2)
$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} R} \arrow[r, two heads] \arrow[dr] & f_* \mathcal{O}_Z = j_* \tilde{f}_* \mathcal{O}_Z \\ & j_* \mathcal{O}_{\operatorname{Spec} R/I} \arrow[u, "\text{has to be surjective too}"'] \end{tikzcd}$$

$\Rightarrow$ $\mathcal{O}_{\operatorname{Spec} R/I} \twoheadrightarrow \tilde{f}_* \mathcal{O}_Z$ $\Rightarrow$ When $\varphi: R \longrightarrow \mathcal{O}_Z(Z)$ is injective.

Step 2. $f(Z) \subseteq X$ closed, X qc $\Rightarrow$ Z qc $\Rightarrow$ $Z = \bigcup_{i=1}^{m} \operatorname{Spec} S_i$ for some rings S_i

$\operatorname{Spec} S_i \subseteq Z$ are open subschemes

Step 3. Claim. $f: Z \longrightarrow X$ is surjective

Known: $f(Z) \subseteq X$ is closed. $\Rightarrow$ Sts $f(Z)$ is dense

↯ $f(Z) \subseteq X$ not dense $\Rightarrow$ $\exists g \in R$: $\emptyset \neq D(g)$ & $D(g) \cap f(Z) = \emptyset$

$\Rightarrow$ $f(Z) \subseteq V(g)$ $\Rightarrow$ $f^\#(g)(p) = 0$ $\forall p \in Z$ $\Rightarrow$ $f^\#(g)|_{\operatorname{Spec} S_i}$ is nilpotent $\forall i$,

i.e. $(f^\#(g)|_{\operatorname{Spec} S_i})^{n_i} = 0$ for some n_i, $(i = 1, \dots, m)$

Let $n := \max_{i=1,\dots,m} n_i$. $\Rightarrow$ $\underbrace{f^\#(g)^n}_{f^\#(g^n)} = 0$ in $\mathcal{O}_Z(Z)$

Since $\varphi = f^\#(\operatorname{Spec} R)$ is injective, $g^n = 0$ $\Rightarrow$ $D(g) = \emptyset$ ↯

Step 4. $f: Z \longrightarrow X = \operatorname{Spec} R$ is a closed immersion

- $|f|$ homeo onto X by Step 3
- $\mathcal{O}_X \twoheadrightarrow f_* \mathcal{O}_Z$ is surjective
- $\varphi = f^\#: R \longrightarrow \mathcal{O}_Z(Z)$ is injective

Need to check: $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Z$ is injective. Then we will have that $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Z$ is an iso, and hence $Z \xrightarrow{\sim} X$.

Let $p \in Z$, $f(p) \in X$.

$$\begin{tikzcd} \mathcal{O}_{X, f(p)} = R_{f(p)} \arrow[r] & \mathcal{O}_{Z,p} \overset{f \text{ homeo}}{=} (f_* \mathcal{O}_Z)_p \\ R \arrow[u, "\text{res}_X"] \arrow[r, hook] & \mathcal{O}_Z(Z) \arrow[u] \end{tikzcd}$$

Let $a \in R$ s.t. $\varphi_{\mathfrak{p}}(\mathrm{res}_X(a)) = 0$. We will show that $\mathrm{res}_X(a) = 0$ in $R_{\mathfrak{p}}$.

This suffices for injectivity of $\varphi_{\mathfrak{p}}$ since $\varphi_{\mathfrak{p}}(\frac{1}{g})$ is a unit $\forall g \in R \setminus f(\mathfrak{p})$.

$\varphi_{\mathfrak{p}}(\mathrm{res}_X(a)) = \mathrm{res}_Z(\varphi(a)) = 0 \Rightarrow \exists$ open $V \subseteq Z$, $\mathfrak{p} \in V$, $\varphi(a)|_V = 0$.

f homeo $\Rightarrow$ we can choose $V := f^{-1}(D(t))$ for some $t \in R$

$= D(\varphi(t))$ since f is a morphism in LRS

$\varphi(a)|_{D(\varphi(t))} = 0$ in $\mathcal{O}_Z(D(\varphi(t)))$

$$\Rightarrow \varphi(a)|_{\underbrace{D(\varphi(t)) \cap \mathrm{Spec}\, S_i}_{\mathrm{Spec}\,(S_i)_{\sigma_i}}} = 0 \in (S_i)_{\sigma_i} \qquad \text{where } \sigma_i := \varphi(t)|_{\mathrm{Spec}\, S_i} \in S_i$$

$$\Rightarrow \exists n_i \geq 1 : \left(\varphi(a) \cdot \varphi(t)^{n_i}\right)|_{\mathrm{Spec}\, S_i} = \varphi(a)|_{\mathrm{Spec}\, S_i} \cdot \sigma_i^{n_i} = 0$$

$$n := \max_{i=1,\dots,m} n_i \Rightarrow \underbrace{\varphi(a)\varphi(t)^n}_{\varphi(at^n)}|_{\mathrm{Spec}\, S_i} = 0 \quad \forall i$$

$$\Rightarrow at^n = 0$$

But $\mathfrak{p} \in V = f^{-1}(D(t)) \Rightarrow f(\mathfrak{p}) \in D(t) \Rightarrow t \in R \setminus f(\mathfrak{p}) \Rightarrow a = 0$ in $R_{f(\mathfrak{p})}$ □

Rmk. In the def of closed subschemes, 1) cannot be dropped:

$Y := \mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \longrightarrow \mathbb{A}^2_{\mathbb{C}}$ This satisfies 2), 3) but not 1)

However, if Y is affine then 1) can be dropped.

Criterion. $f : Y \to X$ closed immersion $\Leftrightarrow$ $\exists$ affine open cover $X = \bigcup_i U_i$ s.t. $f^{-1}(U_i)$ are affine and $\mathcal{O}_X(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ surjective

Pf: $\Rightarrow$: Thm.

$\Leftarrow$: Key Example.

Rmk. a) Compositions of closed immersions are closed immersions.

b) Closed immersions are stable under base change, i.e.

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, hook] \\ X' \arrow[r] & X \end{tikzcd}$$

implies that f is a closed immersion.

(Idea: $B \otimes_A A/_I \cong B/_{IB}$)

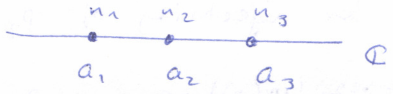
Ex. Closed subschemes of $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\ \mathbb{C}[X]$.

They are of the form $Z = \mathrm{Spec}\ \mathbb{C}[X]/_I$ by Thm.

$0 \neq I = (f)$ for some $f \in \mathbb{C}[X] \setminus \{0\}$. $f = \text{unit} \cdot (x - a_1)^{n_1} \cdots (x - a_k)^{n_k}$

Then $\mathbb{C}[X]/_I = \mathbb{C}[X]/_{(X-a_1)^{n_1}} \times \dots \times \mathbb{C}[X]/_{(X-a_k)^{n_k}}$

$Z = \operatorname{Spec}_Z \mathbb{C}[X]/_I = \coprod_i \operatorname{Spec} \mathbb{C}[X]/_{(X-a_i)^{n_i}}$



$\Rightarrow \{\text{Closed subschemes of } \mathbb{C}\} \longleftrightarrow \{[\mathbb{A}^1_{\mathbb{C}}]\} \amalg \coprod_{n=0}^{\infty} \underbrace{\operatorname{Sym}^n(\mathbb{C})}_{\mathbb{C}^n/_{S_n}}$

Ex. Zero-dimensional subschemes of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[X,Y]$.

Note: for a scheme X, $\dim X = \operatorname{top} \dim X = \sup\{\ell \mid \exists Z_0 \subseteq \dots \subseteq Z_\ell \subseteq X \text{ irred closed}\}$

$\dim \operatorname{Spec} A = \text{Krull} \dim A$

Let $I \subseteq \mathbb{C}[X,Y]$ s.t. $R = \mathbb{C}[X,Y]/_I$ has $\dim R = 0$.

Thm. (CA) R noeth & $\dim R = 0 \iff R$ artinian.

$\Rightarrow \exists$ fin many max ideals $m_1, \dots, m_k$ and $R \xrightarrow{\sim} \underbrace{R/_{m_1^{n_1}}}_{\text{local artinian ring}} \times \dots \times R/_{m_k^{n_k}}$

for some $n_1, \dots, n_k$

$R/_{m_i} \cong \mathbb{C} \Rightarrow \varphi^{-1}(m_i) = (X-a_i, Y-b_i)$ for some $a_i, b_i \in \mathbb{C}$

where $\varphi: \mathbb{C}[X,Y] \twoheadrightarrow R$ is the quot map

Wma: R local artinian, $\bar{m}$ max ideal, $\varphi^{-1}(\bar{m}) = (X,Y) \Rightarrow I \subseteq m = (X,Y)$

Also $\bar{m}^\ell = 0$ for some $\ell \geq 1 \Rightarrow m^\ell \subseteq I$, so choosing I corresponds to choosing $\bar{I} = I/_{m^\ell} \subseteq m/_{m^\ell}$

$d :=$ length of $R = \dim_{\mathbb{C}} R$

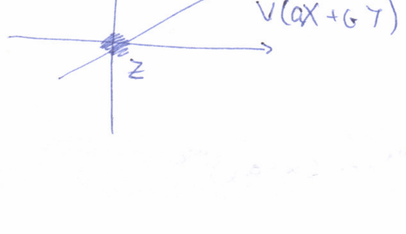
If $d=1$: $R = \mathbb{C}$, $\operatorname{Spec} \mathbb{C} = \{*\}$ reduced pt

If $d=2$: $\underset{2}{R} \supseteq \underset{1}{\bar{m}} \supseteq \underset{0}{\bar{m}} \supseteq \dots$

NAK: $m^n = m^{n+1} \Rightarrow m^n = 0$. Hence $m^2 \subseteq I \subseteq m$

$\Rightarrow I$ corresponds to a subspace $\underbrace{\bar{I}}_{\text{1-dim}} \subseteq m/_{m^2} = \mathbb{C}\langle X\rangle \oplus \mathbb{C}\langle Y\rangle$ Zariski cotangent space

$\Rightarrow I = (\underbrace{X^2, XY, Y^2}_{m^2}, aX+bY)$ for $[a:b] \in \mathbb{P}^1(\mathbb{C})$



$\rightsquigarrow \left\{\begin{array}{c}\text{length 2 0-dim} \\ \text{subschemes of } \mathbb{A}^2_{\mathbb{C}}\end{array}\right\} \xleftrightarrow{1:1} \mathbb{P}^1(\mathbb{C})$

$$I_t = \underbrace{(X,Y)}_{\text{origin}} \cap \underbrace{(X+bt,\ Y-at)}_{(-bt,\ at)} \qquad t \neq 0$$

$$\Big\downarrow t \to 0$$

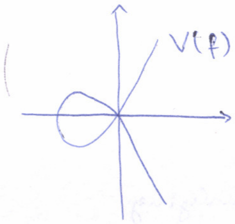
$$I$$

If $d=3$: $R \supseteq \bar{m} \supseteq \bar{m}^2 \supseteq \bar{m}^3 = 0$

Case 1. $\bar{m}^2 = 0 \Rightarrow I = m^2 = (X^2, XY, Y^2)$

Case 2. $I = (Y + \alpha X + \beta X^2, X^3)$ or $(X + \alpha Y + \beta Y^2, Y^3)$

Ex. $f \in \mathbb{C}[x,y]$, $X = V(f)$. What are the 0-dim subschemes of X supp'd at 0?



$f := x \cdot y$

$d=1$: $I = m$

$d=2$: $I = (ax + by, x^2, xy, y^2)$

$$d=3: \quad I = \begin{cases} (x^2, xy, y^2) = m^2 & \sim \{*\} \\ (y + \beta x2, x^3) & \beta \in \mathbb{C} \\ (x + \beta y^2, y^3) & \beta \in \mathbb{C} \end{cases}$$

$\left\{ \begin{array}{l} \text{0-dim subschemes of } X \\ \text{with } d=3 \end{array} \right\} \simeq$;

06.12.2018

Def. X scheme, $\dim X :=$ topdim $X = \sup\{\ell \mid \exists Z_0 \subsetneq \dots \subsetneq Z_\ell \subseteq X$, irred closed$\}$

Ex. $X := \operatorname{Spec} k[x,y,z]/I$, $I = \underbrace{(x)}_{\text{yz-plane}} \cap \underbrace{(y,z)}_{\text{x-axis}} \rightarrow \dim X = 2$

In general, $\dim X = \sup\{\dim X_i \mid X_i \subseteq X$ irred component$\}$ when they exist, e.g. when X is noetherian.

Def. $\operatorname{codim}(Z,X) := \sup\{\ell \mid \exists Z = Z_0 \subsetneq \dots \subsetneq Z_\ell \subseteq X$ irred closed$\}$ for $Z \subseteq X$ irred closed

Prop. $\operatorname{codim}(Z,X) = \dim \mathcal{O}_{X,\eta}$ where η is the generic pt of Z.

Pf: Wma X aff. $\quad Z = Z_0 \subsetneq \dots \subsetneq Z_\ell \subseteq X = \operatorname{Spec} A$

$\longrightarrow \quad \mathfrak{p} = \mathfrak{p}_0 \supsetneq \dots \supsetneq \mathfrak{p}_\ell \supseteq \operatorname{nil} A$

Inside $A_\mathfrak{p} = \mathcal{O}_{X,\eta}$: $\mathfrak{p}A_\mathfrak{p} = \overline{\mathfrak{p}_0} \supsetneq \dots \supsetneq \overline{\mathfrak{p}_\ell}$ where $\eta = \mathfrak{p}$

Thm (Krull PIT) X loc noeth, $f \in \mathcal{O}_X(X) \Rightarrow$ the irred components of $V(f)$ have codim 0 or 1.

Warning. $Z \subseteq X$ irred closed $\Rightarrow \dim Z + \operatorname{codim}(Z,X) \leqq \dim X$, need not be equal

Ex. $Z = pt$, $\dim Z = 0$, $\operatorname{codim}(Z, X) = \dim \mathbb{C}[x]_{(x-a)} = 1$, $\dim X = 2$

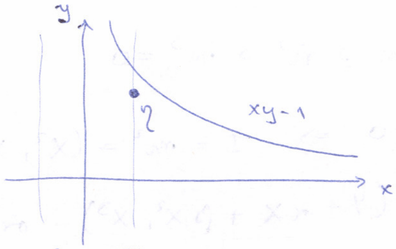
Ex. $A = k[x]_{(x)}[y]$ int domain, $k = \bar{k}$, $0 \subset (x) \subset (x,y)$ chain of primes,

$\Rightarrow \dim A \geq 2$. But A loc of $k[x,y]$ $\Rightarrow$ $\dim A \leq 2$. $\Rightarrow$ $\dim A = 2$.

$f := xy - 1$. $\longrightarrow$ $V(f) = \operatorname{Spec}(A/f) = \operatorname{Spec} k(x)$ as $A/f = k[x]_{(x)}[1/x] = k(x)$

$\Rightarrow \dim V(f) = 0$.

PIT $\Rightarrow$ $\operatorname{codim}(V(f), x) = 1$



Prop. X irred oft $/k$, k not necessarily alg closed. Then
$$\dim Z + \operatorname{codim}(Z, X) = \dim X. \quad \forall Z \subseteq X \text{ irred closed.}$$

Reduced induced subscheme structure

X sch, $Z^{set} \subseteq X$ closed subset

We want to give Z^{set} the structure of a closed subscheme with underlying support Z^{set}.

$Z := (Z^{set}, \mathcal{O}_Z, \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Z)$

$$\{\text{closed subsets of } |X|\} \underset{?}{\overset{|Z| \leftarrow\!\shortmid Z}{\rightleftarrows}} \{\text{closed subschemes of } X\}$$

Easy case. $X = \operatorname{Spec} A$ $\Rightarrow$ $Z^{set} = V(\mathfrak{a})$ for a unique $\mathfrak{a} \subseteq A$ radical ideal

Then let Z be the closed subscheme associated to the closed immersion
$\operatorname{Spec}(A/\mathfrak{a}) \hookrightarrow \operatorname{Spec} A$, i.e. $|Z| = Z^{set}$. This is the red. ind. subsch. assoc. to Z^{set}.

Lemma. In the above situation, let $U = D(g) \subseteq \operatorname{Spec} A$. Then $Z \cap D(g) \subseteq \operatorname{Spec} A_g$ is a closed subscheme, and it is the reduced induced subscheme associated to $Z^{set} \cap D(g)$.

Pf: $Z \cap D(g) \cong \operatorname{Spec}(A/\mathfrak{a})_{\bar{g}} \cong \operatorname{Spec}(A_g/\mathfrak{a}A_g) \cong Z^{set} \cap D(g)$ where $\bar{g}$ is the img of g in $A/\mathfrak{a}$ □

General case. $X = \bigcup U_i$, $U_i = \operatorname{Spec} A_i$, $Z^{set} \subseteq X$ closed subset

Define $Z_i \subseteq U_i$ to be the red ind subsch. assoc to $Z^{set} \cap U_i \subseteq U_i$.

The intersections $U_i \cap U_j$ are covered by affine open sets which are basic opens for both U_i and U_j.

Lemma $\Rightarrow$ $Z_i|_V = \text{riss}(Z^{set} \cap V) = Z_j|_V$ $\rightarrow$ $Z_i|_{U_i \cap U_j} \cong Z_j|_{U_i \cap U_j}$

On multiple intersections, the same argument holds.

Define Z by gluing together the Z_i.

Algebraic Geometry I, lecture 16

Separated morphisms

Recall. X prevar is sep if $\forall Y$ prevar: $\forall\ Y \overset{f}{\underset{g}{\rightrightarrows}} X$: $Eq(f,g) \subseteq Y$ closed, or equivalently, $\Delta(X) \subseteq X \times X$ is closed.

Now let $f: X \to Y$ be a morphism of schemes.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_X"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
& X \underset{Y}{\times} X \arrow[r, "p_2"'] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Def. $f: X \to Y$ is separated if $\Delta_X: X \to X \underset{Y}{\times} X$ is a closed immersion. We also say that X is separated over Y.

Note that Δ_X depends on f.

If $Y = \operatorname{Spec} \mathbb{Z}$, we say that X is separated.

Ex. $X = \operatorname{Spec} A \xrightarrow{f} Y = \operatorname{Spec} B \implies X \underset{Y}{\times} X = \operatorname{Spec} A \underset{B}{\otimes} A$

Δ_X is induced by

$$\begin{aligned} A \underset{B}{\otimes} A &\longrightarrow A \\ a \otimes 1 &\longmapsto a \\ a \otimes a' &\longmapsto aa' \end{aligned}$$

$A \underset{\mathbb{Z}}{\otimes} A \longrightarrow A \underset{B}{\otimes} A \longrightarrow A$ is epi $\implies \Delta_X$ is a closed immersion

Ex. $L := \mathbb{A}^1_k \amalg \mathbb{A}^1_k / D(x) \sim D(y) \implies L \to \operatorname{Spec} k$ is not separated

Prop. $f: X \to Y$ sep $\iff \Delta_X(X) \subseteq X \underset{Y}{\times} X$ is a closed subset.

Pf: • Assume that $\Delta_X(X)$ is closed.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_X"] \arrow[ddr, bend right, "id"'] & \\
& X \underset{Y}{\times} X \arrow[d, "p_1"] \\
& X
\end{tikzcd}$$

$p_1 \big|_{\Delta_X(X)}$ is inverse to $|\Delta_X| \implies \Delta_X$ is a homeomorphism

Let $p \in X$, $\Delta_X(p) \in X \underset{Y}{\times} X$, wts surjectivity of $\mathcal{O}_{X \underset{Y}{\times} X,\, f(p)} \to \mathcal{O}_{X,p}$

Let $V = \operatorname{Spec} B \subseteq Y$ open affine s.t. $f(p) \in V$,

$U = \operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$ open affine s.t. $p \in U$.

Then $U \underset{V}{\times} U \in \Delta_X(p)$ is open, $\Delta_U = \Delta_X|_U: U \to U \underset{V}{\times} U$ is a closed immersion by Ex.

$$\begin{tikzcd}
\mathcal{O}_{X \underset{Y}{\times} X,\, f(p)} \arrow[r] \arrow[d, equal] & \mathcal{O}_{X,p} \arrow[d, equal] \\
\mathcal{O}_{U \underset{V}{\times} U,\, f(p)} \arrow[r, two heads] & \mathcal{O}_{U,p}
\end{tikzcd}$$

$\implies$ (first line) … $\implies f$ is separated.

• The converse is obvious.

Rmk The pf shows that Δ_X is always a locally closed immersion, i.e. it factors as (open immersion)$\circ$(closed immersion), as $X \xhookrightarrow{cl} \bigcup_{U,V} U \underset{V}{\times} U \xhookrightarrow{op} X \underset{Y}{\times} X$.

Ex. Classical interpretation: $a : X \longrightarrow S$

Prop. X sep $/S$ $\Leftrightarrow$ $\forall Y \to S$ $\forall f, g : Y \longrightarrow X$: $Z := \{ y \in Y \mid f \circ i_y = g \circ i_y \} \subseteq Y$ is closed.

$$\begin{tikzcd} Y \arrow[rr] \arrow[rd] & & X \arrow[ld] \\ & S & \end{tikzcd}$$

Lemma. $W \overset{f}{\underset{g}{\rightrightarrows}} X$ over S, $t := f \underset{S}{\times} g : W \longrightarrow X \underset{S}{\times} X$. Then t factors through Δ_X iff $f = g$.

$$\begin{tikzcd} W \arrow[rd, shift left, "t"] \arrow[rd, shift right, "t'"'] \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\ & X \underset{S}{\times} X \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[rd, phantom, "\lrcorner"] & X \arrow[d, "a"'] \\ & X \arrow[r, "a"] & S \end{tikzcd}$$

Pf: $\overset{\Rightarrow}{\bullet}$ If $t = \Delta_X \circ \tilde{t}$ for some $\tilde{t} : W \longrightarrow X$

then $f = p_1 \circ t = \underbrace{p_1 \circ \Delta_X}_{id_X} \circ \tilde{t} = \tilde{t} = \underbrace{p_2 \circ \Delta_X}_{id_X} \circ \tilde{t} = p_2 \circ t = g$

$\overset{\Leftarrow}{\bullet}$ Let $t' := \Delta_X \circ f : W \longrightarrow X \longrightarrow X \underset{S}{\times} X$. Then $\left. \begin{array}{l} p_1 \circ t' = p_1 \circ \Delta_X \circ f = f = p_1 \circ t \\ p_2 \circ t' = p_2 \circ \Delta_X \circ g = g = p_2 \circ t \end{array} \right\} \Rightarrow t = t'$ by univ. prop. □

PF OF PROP: $\overset{\Leftarrow}{\bullet}$ Claim. $\Delta_X(X) = \{ z \in X \underset{S}{\times} X \mid p_1 \circ i_z = p_2 \circ i \}$

PF OF CLAIM: $\supseteq$: by Lemma, as i_Z factors through Δ_X.

$\subseteq$: Let $z \in \Delta_X(X)$. Since $\Delta_X(X)$ is a locally closed immersion, i_Z $\overset{(*)}{\text{factor}}$ as

$$\begin{tikzcd} \operatorname{Spec} k(z) \arrow[r] \arrow[rd] & X \arrow[d, "\Delta_X"] \\ & X \underset{S}{\times} X \end{tikzcd} \Rightarrow z \in RHS \text{ by Lemma.}$$

$(*)$:
$$\begin{tikzcd} z \in Z \arrow[r, hook, "cl"] & U \arrow[r, hook, "op"] & X \\ & \operatorname{Spec} k(z) \arrow[lu, dashed, "(**)"] \arrow[u] \arrow[ru] & \end{tikzcd}$$
$\Rightarrow$ wma X aff, $(Z \to X) = (\operatorname{Spec} A/f \longrightarrow \operatorname{Spec} A)$

$(**)$:
$$\begin{tikzcd} A \arrow[r] \arrow[rd] & A/I \arrow[d, dashed] \\ & k(p) \end{tikzcd}$$
for $z = p \supseteq I$

$\overset{\Rightarrow}{\bullet}$ Claim. $\{ y \in Y \mid f \circ i_x = g \circ i_y \} = (f \underset{S}{\times} g)^{-1} (\Delta_X(X))$

PF OF CLAIM: $\subseteq$: by Lemma, $(f \times g) \circ i_y = (f \circ i_y) \times (g \circ i_y)$ factors through the diagonal

$\supseteq$: let $y \in (f \times g)^{-1} (\Delta_X(X))$.

$$\begin{tikzcd} & y \in X' \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] & \text{loc cl imm} \\ \operatorname{Spec} k(y) \arrow[ru] \arrow[r, "i_Y"'] & Y \arrow[r, "f \times g"'] & X \underset{S}{\times} X & \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \operatorname{Spec} k(y) \arrow[r, "(f \times g) \circ i_Y"] \arrow[d] & X \underset{S}{\times} X \\ X \arrow[r] & X \arrow[u, "\Delta_X"] \end{tikzcd} \Rightarrow f \circ i_y = g \circ i_y \text{ by Lemma.}$$

Cor. If $k = \bar{k}$, X prevar $/k$, $t(X)$ assoc sch $/k$ then X is sep (as prevar) iff $t(X)$ sep $/k$.

Pf: Let $z \in t(X) \times_k t(X)$ be a closed pt.

Then $z \in \Delta_{t(X)}(t(X)) \iff p_1 \circ i_z = p_2 \circ i_z \iff p_1(z) = p_2(z)$

Hence giving $p_1 \circ i_z :$ Spec $k \longrightarrow t(X)$ is equiv to giving $p_1(z) = q$ and an inclusion

$$\begin{tikzcd} k(q) \arrow[r] & k \\ k \arrow[u, hook] \arrow[ur, "id"'] & \end{tikzcd}$$

$\Rightarrow k(q) \simeq k$ canonically

$\Rightarrow p_1 \circ i_z$ is def'd by $p_1(z)$.

$t(X)$ sep $/k \iff \Delta_{t(X)}(\Delta_X(X))$ closed in $t(X) \times_k t(X) \iff \underbrace{\{\text{closed pts in } \Delta_{t(X)}(\Delta_X(X))\}}_{\overset{\parallel}{\{(x,x) \mid x \in X\}}}$ is closed in $X \times X$.

Cor. $k = \bar{k}$. Then $t:$ PreVar $/k \xrightarrow{\text{equiv}}$ int sch oft $/k$

Var $/k \xrightarrow{\text{equiv}}$ sep int sch oft $/k$

Redef. k a field. Then a variety over k is an integral sep'd sch oft $/k$.

Valuation rings

10.12.2018

A int domain, $K = \text{Frac}\, A$

Prop/Def. A is a valuation ring of K if TFEC are satisfied:

1) $\forall x \in K^\times : x \in A$ or $x^{-1} \in A$
2) The set of ideals of A is totally ordered, i.e. Spec $A = \left\{ \begin{array}{l} \bullet\ (0) \\ \bullet\ \mathfrak{p}_1 \\ \bullet\ \mathfrak{p}_2 \\ \vdots \\ \bullet\ \mathfrak{m} \end{array} \right\}$
3) $\exists \Gamma$ totally ordered gp and $v : K^\times \longrightarrow \Gamma$ s.t.

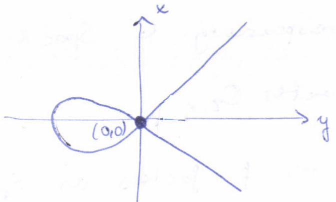
$$A = \{x \in K \mid v(x) \geq 0\} \cup \{0\}.$$

Ex. $A = \mathbb{C}[x]_{(x)}$ valuation ring with $v\left(\frac{p(x)}{q(x)}\right) =$ order of vanishing at $0 = v(p) - v(q)$

where $v(p) = \max\{\ell \mid x^\ell \mid p\}$

Ex. $A = \mathbb{Z}_{(3)}$

Ex. $A = \left(\mathbb{C}[x,y] / (y^2 = x^2(x+1))\right)_{\mathfrak{m}}$



where $\mathfrak{m} = (x,y)$. This is not a val ring. Intuitively: we can't read off the order of fuctions at $(0,0)$, e.g. $f = (x+y)^2 + (x-y)$

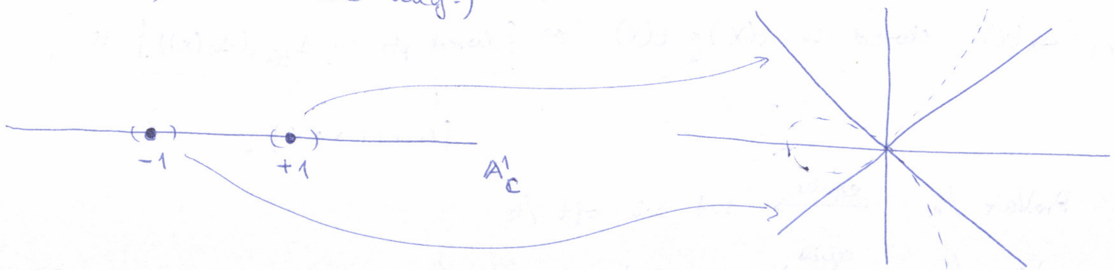
Prop. The valuation rings of K are the max elements of the set of local subrings of K, ordered by domination.

Def. $B, A \subseteq K$ local subrings. Then B dominates A if $A \subseteq B$ & $m_B \cap A = m_A$.

Equivalently: for $f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ induced by $A \hookrightarrow B$, $f(\text{closed pt}) = \text{closed}$.

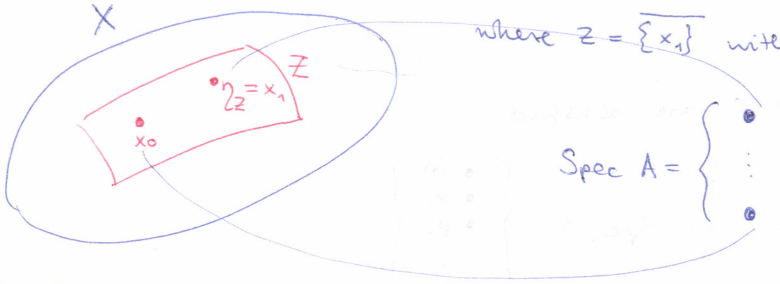
Ex.
$$\underbrace{\mathbb{C}[x,y]/(y^2=x^2(x+1))}_{\mathbb{C}[t^2-1,\, t(t^2-1)]} \hookrightarrow \mathbb{C}[t] \hookrightarrow K = \mathbb{C}(t)$$
$$x \longmapsto t^2-1$$
$$y \longmapsto t(t^2-1)$$

Then $\mathbb{C}[t^2-1, t(t^2-1)]_{(x,y)}$ is dominated by $\mathbb{C}[t]_{(t\pm1)}$, hence not a val ring. ($\mathbb{C}[t]_{(t\pm1)}$ is a val ring.)



Ex. $\mathbb{C}[x,y]_m$ is not a val ring: dominated by $\mathbb{C}[x,y,\frac{x}{y},\frac{x}{y^2},\dots]_{(x,y)}$

Lemma 1. A val ring, K = Frac A. There is a natural bijection between $\operatorname{Hom}_{Sch}(\operatorname{Spec} A, X)$ and $\{(x_0, x_1, k(x_1) \hookrightarrow K) \mid x_0, x_1 \in K,\ x_0 \in \overline{\{x_1\}},\ A \cap k(x_1) \text{ dominates } \mathcal{O}_{Z,x_0}$ where $Z = \overline{\{x_1\}}$ with the reduced induced subscheme structure$\}$



$$\operatorname{Spec} A = \left\{ \begin{matrix} \bullet & (0) \\ & \vdots \\ \bullet & m \end{matrix} \right\}$$

Pf: Given such an $(x_0, x_1, k(x_1) \hookrightarrow K)$, the composition $\mathcal{O}_{Z,x_0} \hookrightarrow A \cap k(x_1) \hookrightarrow A$ is a local homomorphism. Consequently, $f: \operatorname{Spec} A \longrightarrow \operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow Z \xrightarrow{i} X$. is in Hom(--)

Conversely, given $f: \operatorname{Spec} A \longrightarrow X$, let $x_1 := f((0))$, $x_0 := f(m)$. As f is continuous, $x_0 \in \overline{\{x_1\}}$.

$(k(x_1) \hookrightarrow K) :=$ the morph corresponding to $\operatorname{Spec} K \longrightarrow \operatorname{Spec} A \longrightarrow X$

Need to check: $A \cap k(x_1)$ dominates $\mathcal{O}_{Z,x_0}$

$\overline{f(\operatorname{Spec} A)} = Z$, A is reduced $\Rightarrow$ f factors as $\operatorname{Spec} A \xrightarrow{\tilde{f}} Z \longrightarrow X$
$$m \longmapsto x_0$$
by the upcoming lemma 2.

Hence
$$\begin{tikzcd} \mathcal{O}_{Z,x_0} \arrow[r] \arrow[d, "res"] & A_m = A \arrow[d] \\ k(x_1) \arrow[r] & K \end{tikzcd}$$
$\Rightarrow A \cap k(x_1)$ dominates $\mathcal{O}_{Z,x_0}$. □

Lemma 2. T reduced, $f : T \to X$ morph, $Y := \overline{f(T)}$ with riss. Then f factors as
$T \xrightarrow{\tilde{f}} Y \xrightarrow{i} X$

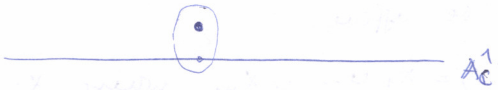
Pf: Wma $X = \operatorname{Spec} B$.

$I := \operatorname{Ker}(B \to \mathcal{O}_T(T))$ reduced ideal in B since T is reduced

$$\begin{tikzcd} B \arrow[rr, "f^{\#}"'] \arrow[dr] & & \mathcal{O}_T(T) \\ & B/I \arrow[ur] & \end{tikzcd} \Rightarrow \begin{tikzcd} \operatorname{Spec} B & & T \arrow[ll] \arrow[dl] \\ & \operatorname{Spec} B/I \arrow[ul] & \end{tikzcd}$$

Valuative criterion for separatedness

Idea. X scheme $/\mathbb{C}$, $\mathbb{A}^1_{\mathbb{C}} \overset{f}{\underset{g}{\rightrightarrows}} X$



$f|_{\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}} = g|_{\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}}$ Does $f = g$? Is the extension to $\mathbb{A}^1_{\mathbb{C}}$ unique?

Yes, if X is sep. The VC is the converse.

Idea. We only care about $f|_{\operatorname{Spec} \mathbb{C}[x]_{(x)}}$ and $g|_{\operatorname{Spec} \mathbb{C}[x]_{(x)}}$

Thm. (VC) $f : X \to Y$ sep, $\Leftrightarrow$ f qs & $\forall A$ val ring with $K = \operatorname{Frac} A$ and

$\forall$ cd: (*)

$$\begin{tikzcd} \operatorname{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A = T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is at most one $T \to X$ making the whole diag commute.

Def. $f : X \to Y$ is quasi-separated $\Leftrightarrow$ $\Delta_X : X \to X \times_Y X$ is qc

Exc. f is qs $\Leftrightarrow$ $\forall$ aff open $\operatorname{Spec} A \subseteq Y$ $\forall U, V \subseteq X$ open affine s.t. $f(U) \subseteq \operatorname{Spec} A$, $f(V) \subseteq \operatorname{Spec} A$:

$U \cap V$ is a finite union of aff opens.

Rmk. If X is loc noeth, every $X \to Y$ is qs.

Thm. (Noetherian version of VC, see Görtz–Wedhorn) Y loc noeth, $f : X \to Y$ oft.

Then f is sep $\Leftrightarrow$ $\forall A$ DVR (the same condition).

Pf OF VC: $\Rightarrow$ f sep $\Leftrightarrow$ Δ_X is a closed immersion $\Rightarrow$ Δ_X is qc $\Rightarrow$ f is qs ✓

Now let A, K, U, T be as above, $\overset{\text{AON}}{f, g} : T \rightrightarrows X$ as in (*).

$Z := \{ t \in T \mid f \circ i_t = g \circ i_t \}$. This is closed since f is sep, and $U \subseteq Z$ by ass.

$\Rightarrow Z = T$. T is reduced $\Rightarrow f = g$ by Exc. 7.4

✓

Lemma 3. $f : X \to Y$ qc. Then $f(X) \subseteq Y$ is closed $\Leftrightarrow$ $f(X)$ is stable under specialisation, i.e. if $x_1 \in f(X)$ and $x_0 \in \overline{\{x_1\}}$ then $x_0 \in f(X)$.

Pf: • $\Rightarrow$ $f(X)$ closed and $x_1 \in f(X) \Rightarrow \overline{\{x_1\}} \subseteq f(X) \Rightarrow x_0 \in f(X)$

• $\Leftarrow$ Rmk. Non-example: $X := \coprod_{a \in \mathbb{C}} \operatorname{Spec} \mathbb{C}[x]/(x-a) \xrightarrow{f} \operatorname{Spec} \mathbb{C}[x] =: Y$

$(0) \notin f(X) = \{(x-a) \mid a \in \mathbb{C}\} \subsetneq \mathbb{A}^1_{\mathbb{C}}$, stable under specialisation but not closed.

Wma X is reduced as $X_{red} \to X$ is a homeo from the underlying space.

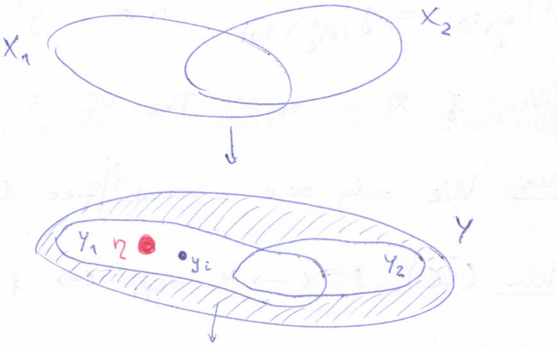
Replace Y by $\overline{f(X)}$ with riss, as $X \to \overline{f(X)} \hookrightarrow Y$ by Lemma 2.

Nts surjectivity. of f.

Assume Y to be affine.

f qc $\Rightarrow f^{-1}(Y) = X_1 \cup \ldots \cup X_m$ where X_i aff opens. Let $Y_i := \overline{f(X_i)}$ with riss.

Let $y \in Y$, $y \in Y_i$ for some i. Let $\eta \in Y_i$ a point s.t. $y \in \overline{\{\eta\}}$ and η is not in the closure of any other point in Y_i



there is nothing here by our assumption $Y = \overline{f(X)}$

Equivalently, let $Y_i = \operatorname{Spec} B$, $y = \mathfrak{p} \in \operatorname{Spec} B$, $\eta = \mathfrak{p}' \in \operatorname{Spec} B$ s.t. $\mathfrak{p}'$ is minimal among all the primes contained in $\mathfrak{p}$.

$X_i = \operatorname{Spec} A$. Then we claim that $\mathfrak{p}' = \eta \in f(X_i) = f(\operatorname{Spec} A)$.

Indeed, consider

$$\begin{tikzcd} A \arrow[r, "\psi"] & A \otimes_B B_i \\ B \arrow[u, "f^\#"] \arrow[r, "\rho"] & B_{\mathfrak{p}'} = k(\mathfrak{p}') \arrow[u, "\varphi"'] \end{tikzcd}$$

Let $\mathfrak{q} \in \operatorname{Spec}(A \otimes_B B_i) \Rightarrow \varphi^{-1}(\mathfrak{q}) = (0)$. Let $\mathfrak{q}' := \psi^{-1}(\mathfrak{q}) \Rightarrow f(\mathfrak{q}') = (f^\#)^{-1}(\mathfrak{q}') = \rho^{-1}((0)) = \mathfrak{p}$.

This shows our claim. Consequently, $y \in f(X)$ by specialisation. ☐

$\Leftarrow$ • Wts $\Delta_X(X)$ is closed. By Lemma 3 and qs-ness of f: sts $\Delta_X(X)$ stable under spec.

Let $x_0, x_1 \in X \times_Y X$ s.t. $x_0 \in \overline{\{x_1\}}$, $x_1 \in \Delta_X(X)$, let $K := k(x_1)$, and let A be the maximal local subring of K dominating $\mathcal{O}_{Z,x_0}$ where $Z = \overline{\{x_1\}}$. Then A is a val-ring.

By Lemma 1:

$$\begin{tikzcd} U = \operatorname{Spec} K \arrow[r] \arrow[d, "j"'] & X \arrow[d, "\Delta_X"] \\ T = \operatorname{Spec} A \arrow[r, "t"] & X \times_Y X \end{tikzcd}$$

We get the upper horis. arrow since $x_1 \in \Delta_X(X)$ and Δ_X is a loc closed imm.

AON

$f := p_1 \circ t$, $g := p_2 \circ t$: $\operatorname{Spec} A \rightrightarrows X$ (over Y) where p_1, p_2 are the projections

$f|_U = (p_1 \circ t \circ j) = p_1 \circ \Delta_X \circ \tilde{t} = p_2 \circ \Delta_X \circ \tilde{t} = g|_U$

By assumption, $f = g \Rightarrow t$ factors through $\Delta_X \Rightarrow t(m) = x_0 \in \Delta_X(X)$

☐

Algebraic Geometry I, lecture 17-18

Rmk. Open or closed immersions are separated.

The property of being separated is preserved under composition, base change, and is local on target.

If $X \to Y \to Z$ is sep then so is $X \to Y$. $(*)$

Def. Local on target: $\exists\, Y = \bigcup U_i$ open cover s.t. $f|_{f^{-1}(U_i)}$ has the property.
$f : X \to Y$

13.12.2018

Pf of $(*)$ when X is loc noeth:

X loc noeth $\Rightarrow$ $X \to Y$ qs. We only need to check the val crit.

$$\begin{tikzcd}
U = \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
T = \operatorname{Spec} A \arrow[r] \arrow[ur, bend left, "a"] \arrow[ur, bend right, "b"'] \arrow[dr] & Y \arrow[d] \\
 & Z
\end{tikzcd}$$

Need to check: $a = b$.

Apply VC to $X \to Z$ ($X \to Z$ is sep) $\Rightarrow a = b$.

Another Pf: let $S := Z$. Factor f as

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f = \mathrm{id}_X \times_S f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times_S Y \arrow[r, "p_2"] \arrow[d, "f \times_S \mathrm{id}_Y"] \arrow[dr, "p_1"] \arrow[r, phantom, "\lrcorner"'] & Y \arrow[dr] & \\
Y \arrow[r, "\Delta_Y"'] & Y \times_S Y & X \arrow[r, "g"'] & S
\end{tikzcd}$$

Δ_Y loc cl imm $\Rightarrow$ Γ_f closed imm.

$\Rightarrow$ Γ_f separated

g is sep by assumption $\Rightarrow$ p_2 sep

$\Rightarrow f = p_2 \circ \Gamma_f$ is separated

Proper morphisms

Property	Topological analogue	Interpretation
separated	T2	uniqueness of limits
univ. closed	qc	existence of limits
proper	cpt = qc + T2	uniqueness + existence of limits

Def. $f : X \to Y$ univ closed if $\forall\, Y' \to Y$: the base change $f' : X' = X \times_Y Y' \to Y'$ is closed.

Def. $f : X \to Y$ is proper if separated, oft and univ closed.

Ex. $X = \mathbb{A}^1_k \to k$ not proper:

$$\begin{tikzcd}
\mathbb{A}^2_k \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & X = \mathbb{A}^1_k \arrow[d] \\
\mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

p is not closed since $p(V(xy-1)) = D(x)$

Ex. Finite morphisms are proper. E.g. $\mathrm{Spec}\,\mathbb{C}[x,y]/_{y^2=x^3-x}$ is proper $/\mathrm{Spec}\,\mathbb{C}[x]$

Thm. (VC for properness) $f: X \to Y$, TFAE:

(1) f proper

(2) f qs, oft, and $\forall$ A val rings and $\forall$ cd

$$\begin{tikzcd}
U = \mathrm{Spec}\,K \arrow[r] \arrow[d] & X \arrow[d] \\
T = \mathrm{Spec}\,A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y
\end{tikzcd}$$

where $K = \mathrm{Frac}\,A$ $\exists!$ unique lift $T \to X$.

Pf: $\overset{\Rightarrow}{\cdot}$ Let f be proper. In particular, qs (because sep) and oft.

Let A be any valuation ring, and let us have a cd as above.

$$\begin{tikzcd}
\mathrm{Spec}\,K \arrow[r, "a"] \arrow[dr, bend right] & X' \arrow[r] \arrow[d, "p"] & X \arrow[d] \\
& T = \mathrm{Spec}\,A \arrow[r] \arrow[u, dashed, bend left, "g"] & Y
\end{tikzcd}$$

$\xi_1 := a((0)) \in X'$, $Z := \overline{\{\xi_1\}}$ with riss. Then $p(Z)$ is closed in T and $(0) \in p(Z)$
$\Rightarrow p(Z) = \mathrm{Spec}\,A$.

Let $\xi_0 \in Z$ s.t. $p(\xi_0) = \mathfrak{m} \in \mathrm{Spec}\,A$ closed point.

$$\begin{tikzcd}
A_{\mathfrak{m}} \arrow[r, hook, "\text{local}", "(p|_Z)^\#_{\xi_0}"'] \arrow[d] & \mathcal{O}_{Z,\xi_0} \arrow[d, "\text{res}"] \arrow[l, dashed, bend left, "\psi"] & \\
K \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id} = (p|_Z \circ a)^\#"'] & \mathcal{O}_{Z,\xi_1} \underset{\substack{\uparrow \\ Z \text{ is integral}}}{=} \mathrm{Frac}\,\mathcal{O}_{Z,\xi_0} \arrow[r, "a^\#"] & K
\end{tikzcd}$$

$\Rightarrow \mathcal{O}_{Z,\xi_1} = K$ and $\mathcal{O}_{Z,\xi_0}$ dominates A

$\Rightarrow \mathcal{O}_{Z,\xi_0} = A$ since A is a val rg $\Rightarrow \exists \psi$ inverse

Let g: $\mathrm{Spec}\,A \to \mathrm{Spec}\,\mathcal{O}_{Z,\xi_0} \to Z \to X$ be the morphism induced by $\psi : \mathcal{O}_{Z,\xi_0} \to A$.

By construction, g is a section of p. $\Rightarrow$ $(X' \to X) \circ g$ is the desired morphism $T \to X$. Uniqueness?

$\overset{\Leftarrow}{\cdot}$ Let f satisfy VC. We show that f is univ closed, i.e. $\forall Y' \to Y$:

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

f' is closed.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X' \arrow[r, hook] \arrow[d, "f'"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "g'"] \arrow[urr, "\exists! g"'] & Y' \arrow[r] & Y \end{tikzcd}$$

VC $\Rightarrow$ $\exists!\ g : T \to X$

Univ prop $\Rightarrow$ $\exists!\ g' : T \to X'$

We show that f' is closed. Let $Z \subseteq X'$ be a closed subset with riss.

We show that $f'(Z)$ is stable under specialisation.

Let $x_0, x_1 \in Y'$ s.t. $x_0 \in \overline{\{x_1\}}$, $x_1 \in f'(Z)$

Let $\xi_1 \in Z$ s.t. $f'(\xi_1) = x_1$

Let $A' := \mathcal{O}_{\overline{\{x_1\}}, x_0} \subseteq k(x_1) = \mathcal{O}_{\overline{\{x_1\}}, x_1}$

Let $A \subseteq k(\xi_1) = \mathcal{O}_{\overline{\{\xi_1\}}, \xi_1}$ be a val ring dominating A'. via $(f')^{\#} : k(x_1) \hookrightarrow k(\xi_1)$

$$\begin{tikzcd} K = k(\xi_1) & k(x_1) \arrow[l, hook'] \\ A \arrow[u, hook] & \mathcal{O}_{\overline{\{x_1\}}, x_0} \arrow[u] \arrow[l, "loc"] \end{tikzcd} \Rightarrow \begin{tikzcd} \mathrm{Spec}\, K \arrow[r, "i_{\xi_1}"] \arrow[d] & X' \arrow[d, "f'"] \\ \mathrm{Spec}\, A \arrow[r] \arrow[ur, "\exists g"] & Y' \\ m \arrow[r, mapsto] & x_0 \\ (0) \arrow[r, mapsto] & x_1 \end{tikzcd}$$

VC $\Rightarrow$ $\exists g$ $\Rightarrow$ $x_0 = f'(g(m)) \in f'(Z)$

Lemma. A: ring but not a field. TFAE:

(1) A noeth val ring

(2) A local 1-dimensional integrally closed noetherian domain

(3) A local PID

(4) A integral domain and $\exists v : K^\times \longrightarrow \mathbb{Z}$, $v(ab) = v(a) + v(b)$, $v(a+b) \geq \min(v(a), v(b))$ s.t. $A = v(K^\times) \cup \{0\}$ where $K = \mathrm{Frac}\, A$.

Def. Such an A is a DVR.

Rmk. $f : X \to Y$, Y loc noeth. $\Rightarrow$ it suffices to check the VC for DVRs.

Ex. Let C be a proper curve $/k$. Also assume that C is normal.

Def. Curve: variety of dimension 1.

Let $\eta \in C$ be its gen pt., $K := k(\eta) = \mathcal{O}_{C,\eta}$ function field of C

Let $x \in C$ be a closed pt.

Then $\mathcal{O}_{C,x}$ is local and of dim 1 since $\dim C = 1$, $\dim\{x\} = 0$ and C is oft, hence $\underbrace{\mathrm{codim}(\{x\}, C)}_{\dim \mathcal{O}_{C,x}} = \dim C - \dim\{x\}$. $\mathcal{O}_{C,x}$ is also noetherian and int closed.

$\Rightarrow \mathcal{O}_{C,x}$ is a DVR. of K/k

Conversely, let A be a DVR of K/k, where $K = \operatorname{Frac} A$ and $k \subseteq A$.

Consider

$$\begin{tikzcd}
U = \operatorname{Spec} K \arrow[r, "i_\eta"] \arrow[d, hook] & C \arrow[d] \\
T = \operatorname{Spec} A \arrow[r] \arrow[ur, "\exists! h"] & \operatorname{Spec} k
\end{tikzcd}$$

VC for C : $\exists!$ h lift

let $x := f_A(m)$

$(f_A^\#)_m : \mathcal{O}_{C,x} \xhookrightarrow{loc} A \longrightarrow K$

In part., $\mathcal{O}_{C,x} \subset A \subset K$ and $\mathcal{O}_{C,x}$ and A are DVRs. $\Rightarrow A = \mathcal{O}_{C,x}$

$$\Rightarrow \quad \{\text{closed pts of } C\} \xleftrightarrow{1:1} \{\text{DVR of } K/k\}$$

$$x \longmapsto \mathcal{O}_{C,x}$$

<u>Question</u>. $\{\text{normal proper curves}/k\} \underset{\xleftarrow{\Psi}}{\longrightarrow} \{\text{function fields of dim } 1\}_{/k}$

$$C \longmapsto K(C)$$

Is there an inverse map?

<u>Def</u>. A function field of dim 1 over k is a fin gen field extn K/k of trdeg 1.

<u>Claim</u>. $\exists!\ \Psi$.

Pf: Let K be a funct field of dim 1, $C = C_K$ s.t. $K(C) = K$.

$C := \{\text{valuation rings of } K/k\} = \{\text{DVRs of } K/k\} \amalg \underbrace{\{[K]\}}_{=:\eta}$ as a set

For $p \in C$ write $A_p \subseteq K$ for the associated val ring.

Topology on C: let $Z \subseteq C$ be closed if $Z = C$ or Z is a finite subset of $C \setminus \{\eta\}$

Structure sheaf: $\mathcal{O}_C(U) := \bigcap_{p \in C} A_p \subseteq K$

We show that $(C, \mathcal{O}_C)$ is a scheme.

$\mathcal{O}_{C,p} = \varinjlim_{p \in U} \bigcap_{q \in U} A_q = A$ local ring $\rightarrow (C, \mathcal{O}_C) \in \operatorname{Ob} \mathrm{LRS}$

Nts there is an aff open cover. For $x \in K^\times$, let $U_x := \{p \in C \mid x \in A_p\}$

Then U_x is open: $p \in C \setminus U_x \Leftrightarrow x \notin A_p \Leftrightarrow \frac{1}{x} \in m_p \subseteq A_p$

$y := 1/x$, $R :=$ int closure of $k[y] \subset K$

$K/k(y)$ is fin $\Rightarrow$ R is a fin gen k-algebra

$\dim R = \operatorname{trdeg} \operatorname{Frac} R = \operatorname{trdeg} K = 1 \quad \Rightarrow R_q$ is a DVR $\forall q \in (\operatorname{Spec} R) \setminus \{(0)\}$

$U_y \longrightarrow \operatorname{Spec} R$ is a bijection: if $p \in C$ closed pt s.t. $y \in A_p$

$p \longmapsto m_p \cap R$ then $k[y] \subseteq A_p \Rightarrow R \subseteq A_p \Rightarrow R_{m_p \cap R} \subseteq A_p$ DVR

$\Rightarrow R_{m_p \cap R} = A_p$.

$\Rightarrow C \setminus U_x = \{p \in U_y \mid y \in m_p\} \simeq \{q \in \operatorname{Spec} R \mid y(q) = 0\} = V(y)$

$V(y)$ is finite since $\dim R = 1$ and PIT. ✓

We show that $(U_x, \mathcal{O}_C|_{U_x})$ is affine:

$R :=$ int cl of $k[x]$ in K $\Rightarrow$ $U_x \cong \operatorname{Spec} R$ homeo.

$$\mathcal{O}_{\operatorname{Spec} R}(V) = \bigcap_{x \in V} \mathcal{O}_{\operatorname{Spec} R, x} = \bigcap_{x \in V} A_x = \mathcal{O}_C(V)$$

These U_x cover C.

$\Rightarrow (C, \mathcal{O}_C)$ is a scheme.

Ex. $K = \mathbb{C}(t)$ over $k = \mathbb{C}$, $A \subset K$ DVR

Case 1: $t \in A \Rightarrow \mathbb{C}[t] \underset{\text{int cl}}{\subset} A \subset \mathbb{C}(t) \Rightarrow A = \mathbb{C}[t]_{(t-a)}$ for some $a \in \mathbb{C}$

Case 2: $\frac{1}{t} \in A \Rightarrow A = \mathbb{C}[u]_{(u-b)}$, $b \in \mathbb{C}$

If $a \neq 0$ then $\mathbb{C}[t]_{(t-a)} = \mathbb{C}[u]_{(u-\frac{1}{a})}$

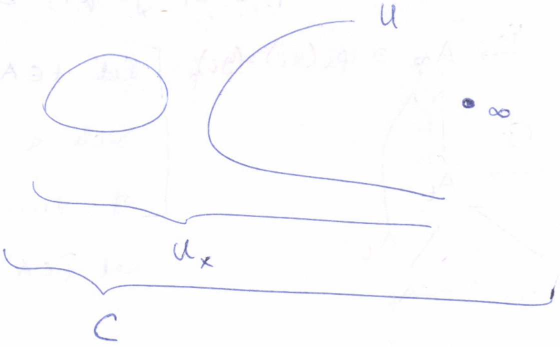
$\Rightarrow C = U_t \amalg U_u$ where $U_t = \operatorname{Spec} \mathbb{C}[t]$, $U_u = \operatorname{Spec} \mathbb{C}[u]$

$\varphi : U_t \setminus \{0\} \longrightarrow U_u \setminus \{0\}$

$t = \frac{1}{u} \longmapsto u$

Ex. $K = \mathbb{C}(x, \underbrace{\sqrt{x^3 - x}}_{y})$ i.e. $y^2 = x^3 - x$

$U_x = \operatorname{Spec} \mathbb{C}[x,y] / y^2 - x^3 + x$



Recall. C normal proper curve $/k$ $\rightsquigarrow K = K(C) = \mathcal{O}_{C,\eta}$ 17.12.2018

Conversely: K function field of dim 1 $\rightsquigarrow$ C scheme.

Ex. $K = \mathbb{C}(x, \sqrt{x^3 - x})$, $k = \mathbb{C}$

$C_K = \{\text{DVRs of } K/k\} \amalg \underbrace{\{K\}}_{2}$

$t \in K$: $U_t = \{p \in C_K \mid t \in A_p\}$ where A_p is the val ring corresponding to p

$\forall A \subsetneq K$: $x \in A$ or $\frac{1}{x} \in A \Rightarrow U_x \cup U_{\frac{1}{x}} = C$. Now let $\underline{x}$ be fixed, and compute U_x:

$R :=$ int cl of $k[x]$ in K

$y := \sqrt{x^3 - x} \longrightarrow y^2 = x^3 - x \longrightarrow R = k[x,y] / y^2 - x^3 + x$, $U_x = \operatorname{Spec} R$

Computing $U_{\frac{1}{x}}$: $t := \frac{1}{x}$, $S :=$ int cl of $k[t]$ in K, $u := \frac{y}{x^2} \longrightarrow u^2 = \frac{1}{x} - \frac{1}{x^3} = t - t^3$

$\Rightarrow S = k[u] / u^2 - t - t^3$. $\Rightarrow C = \operatorname{Spec} R \cup \operatorname{Spec} S$.

Lemma. $C = C_K$ is a normal proper curve $/k$.

Pf: Since C is covered by U_x, $U_{\frac{1}{x}}$ and $k[t] \subset K$, it is oft by the following:

$R =$ int cl $k[t]$ in K, $K/K(t)$ fin $\Rightarrow$ R oft $/k$.

$\mathcal{O}_{C,p} = A_p$ is int closed by def $\Rightarrow$ normality

To check properness, use VC. $\square$

Thm. Contravar equivalence of cats b/w

- cat of proper normal curves $/k$, non-constant morphisms
- cat of function fields of dim 1.

given by Φ: $C \longmapsto K(C)$

$(f: C \to C') \longmapsto (f^\#_\eta : K(C') \longrightarrow K(C))$

Note that $f^\#_\eta$ is well-def'd since $\eta \mapsto \eta'$ because f is non-constant.

Lemma^1. X, Y sch $/k$, $x \in X$, $y \in Y$, Y oft, $\varphi_x : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$.

Then $\exists\, U \ni x$ open and $f: U \to Y$ s.t. $f(x) = y$ and $(f^\#_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}) = \varphi_x$.

Pf: Wma X, Y to be affine by shrinking.

$X = \operatorname{Spec} A$, $x = p$, $Y = \operatorname{Spec} B$, $y = q$, $B = k[x_1, \dots, x_n]/(g_1, \dots, g_m)$,

$$\begin{tikzcd}
B_q \arrow[r, "\varphi_x"] & A_p \ni \varphi_x(x_i) = (y_i)_p \\
B \arrow[u] \arrow[r] & A_{t,\tilde t} \arrow[u] \\
x_i \in k[x_1, \dots, x_n] \arrow[u] \arrow[r] & A_t \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

Let $t \in A$ s.t. $\exists\, y_1, \dots, y_n \in A_t$, $\varphi_x(x_i) = (y_i)_p$.

Such a t exists, e.g. product of denominators of $g_1, \dots, g_m$.

Let $\tilde t \in A$ s.t. $g_i(y_1, \dots, y_n) = 0$ in $A_{t,\tilde t}$ $\forall i$

$U \subseteq X$ open $\Rightarrow$ $U \in \operatorname{Spec} A_{t,\tilde t} \longrightarrow \operatorname{Spec} B = Y$ induced by $B \longrightarrow A_{t,\tilde t}$ $\square$

PF OF THM: Ψ: $K \longmapsto C_K$ on objects.

On morphisms: let $\varphi: K' \longrightarrow K$, and define $\Psi(\varphi): C \to C'$, where $C = C_K$, $C' = C_{K'}$, by applying the Lemma^1 to $\eta \in C$, $\eta' \in C'$, $f: K' \longrightarrow K$ $\Rightarrow$ $\exists\, U \subseteq C$ nonempty open and $f: U \longrightarrow C'$ s.t. $f_\eta = \varphi$ $\Rightarrow$ $U = C \setminus \{p_1, \dots, p_n\}$.

Claim. f extends uniquely to $\bar f: C \longrightarrow C'$

Assuming the Claim, set $\Psi(\varphi) := \bar f$. $\Rightarrow$ $\Phi(\Psi(\varphi)) = \varphi$, $\Psi(\Phi(f)) = f$ are easily checked using sep'ness of C.

<u>Lemma</u>^{2} C normal curve $/k$, X proper scheme $/k$, $p \in C$ closed pt.

Then any $f : C \setminus \{p\} \to X$ extends uniquely to a morphism $C \to X$.

Using the Lemma 2 n times, the Claim follows.

PF OF LEMMA^2:

$K = K(C)$

$$\begin{tikzcd}
\mathrm{Spec}\, K \arrow[r, hook, "\text{incl of } \eta"] \arrow[d] & C \setminus \{p\} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\
\underbrace{\mathrm{Spec}\, \mathcal{O}_{C,p}}_{\text{DVR by assumption}} \arrow[r] \arrow[urr, dashed, "\exists!\, g", "VC"'] & C \arrow[r] & \mathrm{Spec}\, k \\
& V \arrow[u, phantom, "\cup" rotated] \arrow[uur, dashed, "h"'] &
\end{tikzcd}$$

X proper $\to$ $\exists!\, g$ by VC

Let $(g^\#)_p : \mathcal{O}_{X,q} \longrightarrow \mathcal{O}_{C,p}$ where $q := g(p)$ be the induced morphism, $p = \mathfrak{m}_p \in \mathrm{Spec}\, \mathcal{O}_{C,p}$

Lemma 1 $\Rightarrow$ $\exists\, V \subseteq C$ open, $p \in V$, $h : V \longrightarrow X$: $g^\#_p = h^\#_p$

By construction: $h|_{\mathrm{Spec}\, K} = g|_{\mathrm{Spec}\, K} = f|_{\mathrm{Spec}\, K} \Rightarrow f|_W = h|_W$ for some $W \neq \emptyset$ open, $W \subseteq V \setminus \{p\} \subseteq X$ $\Rightarrow$ $h|_{V \setminus \{p\}} = f|_{V \setminus \{p\}}$ by sep'ness of X

Let $\bar{f} : C \to X$ be obtained by gluing g and f along $V \setminus \{p\}$.

This $\bar{f}$ is unique since X is sep'd.

Projective spectrum

20.12.2018

<u>Def</u>. <u>Graded ring</u>: $R = \bigoplus_{d=0}^{\infty} R_d$ decomposition into ab gps, $R_d \cdot R_e \subseteq R_{d+e}$ $\forall d, e \geq 0$.

An elt $f \in R_d$ is called <u>homogeneous of degree d</u>.

<u>Ex.</u> $k[x_0, \dots, x_n]$; $\deg x_i = a_i \geq 0$, usually $\forall \deg x_i = 1$.

<u>Ex.</u> $I \subseteq A$ homog ideal. of a gr rg A $\Rightarrow$ A/I gr rg too, $(A/I)_d = A_d/I_d$, $I_d = I \cap A_d$.

In fact, $\bigoplus_{d=0}^{\infty} A_d/I_d \twoheadrightarrow A/I$ is injective iff I is homogeneous.

<u>Ex.</u> $f \in A$ homogeneous $\Rightarrow$ A_f is graded, $(A_f)_d = \left\{ \frac{a}{f^\ell} \,\middle|\, a \in A \text{ homog}, \deg a - \ell \deg f = d \right\}$

$A_f = \bigoplus_{d \in \mathbb{Z}} (A_f)_d$ $\mathbb{Z}$-graded ring.

Graded localisation: <u>$A_{(f)}$</u> $:= (A_f)_0$ Beware of this notation!

<u>Ex.</u> $A = k[x_0, \dots, x_n]$, $\deg x_i = 1$ (this is always how the degrees are given unless stated otherwise)

$(A_{x_i})_0 = \left(k[x_0, \dots, x_n, \tfrac{1}{x_i}]\right)_0 = k\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}\right]$.

Proj construction

A a graded ring, $A_+ := \bigoplus_{d \geq 1} A_d$ augmentation ideal

$\mathrm{Proj}\, A = \{ p \subseteq A \mid p \text{ homogeneous prime ideal, } p \not\supseteq A_+ \}$

Ex.1. $A = k[X]$, $\deg X = 1$, $A_+ = (X) \Rightarrow \mathrm{Proj}\, A = \{(0)\}$

Ex.0. $A = A_0$, $A_+ = (0) \Rightarrow \mathrm{Proj}\, A = \emptyset$

Ex.2. $A = k[x,y]$, $k = \bar{k}$, $A_+ = (x,y)$

$$\mathrm{Spec}\, A = \underset{\substack{\uparrow \\ \text{homog.}}}{\{(0)\}} \amalg \underset{\substack{\uparrow \\ \text{homog. iff } f \text{ is homog.}}}{\{(f) \mid f \in k[x,y] \text{ irred}\}} \amalg \underset{\substack{\uparrow \\ \text{homog iff } a=b=0}}{\{(x-a, y-b) \mid a,b \in k\}}$$

Note that $(x,y) \notin \mathrm{Proj}$ b/c $(x,y) \supseteq A_+$.

$\Rightarrow \mathrm{Proj}\, A = \{(0)\} \amalg \{(f) \mid f \in k[x,y] \text{ irred homog.}\}$

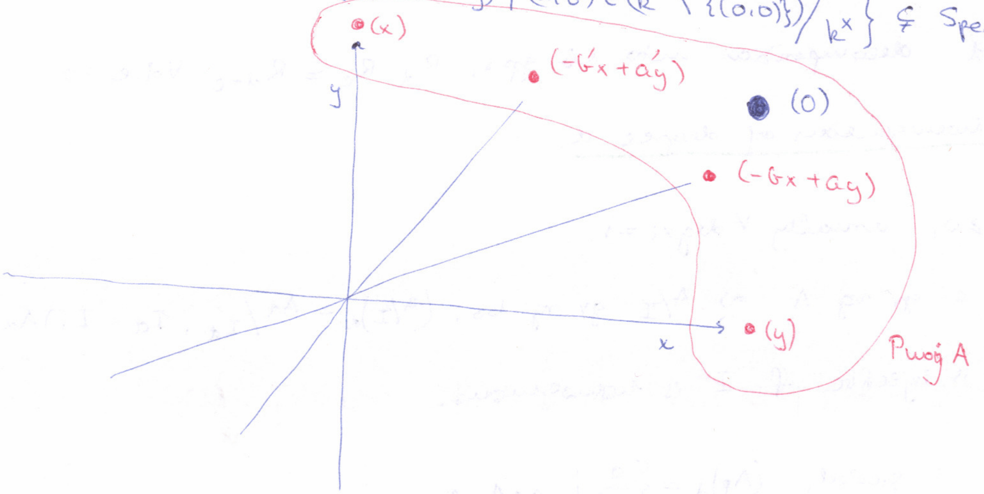
What are the irred homog elts of $k[x,y]$?

Lemma. Let $0 \neq f \in k[x,y]$. Then f is irreducible + homogeneous $\Leftrightarrow f = a_0 x + a_1 y$ for some $a_0, a_1 \in k^2 \setminus \{(0,0)\}$. (We still assume $k = \bar{k}$.)

Pf: Let f be homog + irred. $\Rightarrow f = a_0 x^d + \ldots + a_{d-1} x y^{d-1} + a_d y^d$

$\Rightarrow \hat{f} = a_0 \left(\frac{x}{y}\right)^d + a_1 \left(\frac{x}{y}\right)^{d-1} + \ldots + a_d$ irred in $k[\frac{x}{y}] \Rightarrow$ must be of $\deg \leq 1$. □

$\Rightarrow \mathrm{Proj}\, A = \{(0)\} \amalg \{(-bx + ay) \mid (a,b) \in (k^2 \setminus \{(0,0)\}) / k^\times\} \subsetneq \mathrm{Spec}\, A$



$\mathrm{Spec}\, A = \mathbb{A}^2_k$

Ex 3. $A = \mathbb{R}[x,y] \to \mathrm{Proj}\, A = \{(0)\} \amalg \{(-bx + ay) \mid (a,b) \in (\mathbb{R}^2 \setminus \{(0,0)\}) / \mathbb{R}^\times\}$

$\amalg \{(-bx + ay)(-\bar{b}x + \bar{a}y) \mid (a,b) \in (\mathbb{C} \setminus \mathbb{R})^2 / \mathbb{R}^\times\}$

Ex 4. $A = k[x,y,z]$, $k = \bar{k}$

$$\mathrm{Spec}\, k[x,y,z] = \underset{\substack{\nearrow \\ \text{homog.}}}{\{(0)\}} \amalg \underset{\substack{\nearrow \\ \text{homog iff } f \text{ is homog.}}}{\{(f) \mid f \in A \text{ irr'ble}\}} \amalg \underset{\substack{\nearrow \\ ?}}{\{p \subseteq k[x,y,z] \mid \mathrm{codim}\, p = 2\}}$$

$\amalg \{(x-a, y-b, z-c) \mid a,b,c \in k\}$

Algebraic Geometry 1, lecture 20

Lemma. $\mathfrak{p}$ is homog. $\Leftrightarrow$ $\mathfrak{p} = (bx - ay,\ cx - az,\ cy - bz) =: \mathfrak{p}_{a,b,c}$

Pf: (Not entirely rigorous.)

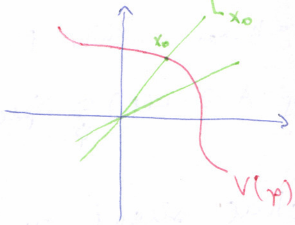
If $\mathfrak{p}$ homog then $\mathfrak{p} = (f_1, \dots, f_l)$ for f_i homog polynomials

$\Rightarrow f_i(t\underline{x}) = t^{\deg f_i} \cdot f_i(\underline{x}) \quad \forall t \in k$

$\underline{x} \in V(\mathfrak{p}) \Leftrightarrow f_i(\underline{x}) = 0 \quad \forall i$

$\Rightarrow f_i(t\underline{x}) = 0 \quad \forall i \ \forall t$

$\Rightarrow t\underline{x} \in V(\mathfrak{p}) \quad \forall t$



Now let $x_0 \in V(\mathfrak{p})$ be a closed pt.

$L_{x_0} := \{t x_0 \mid t \in k\} \subseteq V(\mathfrak{p})$. Since $V(\mathfrak{p})$ is closed & irred, $V(\mathfrak{p}) = L_{x_0}$.

$\Rightarrow \operatorname{Proj} k[x,y,z] = \{(0)\} \sqcup \{(f) \mid f \in k[x,y,z] \text{ irred}\} \sqcup \{\mathfrak{p}_{a,b,c} \mid a,b,c \in (k^3 \setminus \{0\})/_{k^\times}\}$

Proj as a scheme

$V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ for $\mathfrak{a} \subseteq A$ any homog ideal

Lemma. $\{V(\mathfrak{a}) \mid \mathfrak{a} \subseteq A \text{ homog}\}$ form the closed subsets of a topology on $\operatorname{Proj} A$, called the Zariski topology.

$D(f) := \{\mathfrak{p} \in \operatorname{Proj} A \mid f \notin \mathfrak{p}\}$

Lemma. $\{D(f) \mid f \in A\}$ form a basis of the Zariski topology.

Consider the localisation map $A \xrightarrow{\varphi} A_f \supseteq (A_f)_0$.

Prop: $\psi: D(f) \longrightarrow \operatorname{Spec}(A_f)_0$

$\mathfrak{p} \longmapsto \mathfrak{p}A_f \longmapsto (\mathfrak{p}A_f)_0 = \mathfrak{p}A_f \cap (A_f)_0$ is a homeomorphism.

Pf: Let $\mathfrak{q} \in \operatorname{Spec}(A_f)_0$, and define $\tilde{\mathfrak{q}}_d := \left\{ a \in (A_f)_d \;\middle|\; \underbrace{\frac{a^{\deg f}}{f^d}}_{\deg = 0} \in \mathfrak{q} \right\}$

$\tilde{\mathfrak{q}} := \bigoplus_{d \in \mathbb{Z}} \tilde{\mathfrak{q}}_d \subseteq A_f$ homogeneous prime ideal

$\vartheta: \operatorname{Spec}(A_f)_0 \longrightarrow D(f)$

$\mathfrak{q} \longmapsto \varphi^{-1}(\tilde{\mathfrak{q}})$ We claim that ψ and ϑ are inverse to each other.

$\psi(\vartheta(\mathfrak{q})) = \tilde{\mathfrak{q}} \cap (A_f)_0 = \left\{ a \in (A_f)_0 \;\middle|\; \frac{a^{\deg f}}{f^d} \in \mathfrak{q} \right\} = \{ a \in (A_f)_0 \mid a \in \mathfrak{q} \} = \mathfrak{q}$ ✓

$\vartheta(\psi(\mathfrak{p})) \supseteq \mathfrak{p}$: if $a \in \mathfrak{p}_d$ then $\frac{a^{\deg f}}{f^d} \in \mathfrak{q} = \psi(\mathfrak{p}) \Rightarrow a \in \vartheta(\psi(\mathfrak{p}))$ ✓

$\vartheta(\psi(\mathfrak{p})) \subseteq \mathfrak{p}$: if $a \in A_d$ and $\frac{a^{\deg f}}{f^d} \in \mathfrak{q} \in \psi(\mathfrak{p}) \Rightarrow a^{\deg f} \in \mathfrak{p}A_f \Rightarrow a \in \mathfrak{p}A_f \Rightarrow a \in \mathfrak{p}$. ✓

Hence ψ and ϑ are bijections inverse to each other.

If $g \in A$ homog and $D(g) \subseteq D(f)$ then

$$\mathfrak{p} \in D(g) \iff g \notin \mathfrak{p} \iff g \notin \mathfrak{p}A_f \iff \underbrace{\frac{g^{\deg f}}{f^{\deg g}}}_{\bar{g}} \notin (\mathfrak{p}A_f)_0$$

$\Rightarrow \psi(D(g)) = D(\bar{g}) \subseteq \operatorname{Spec}(A_f)_0$. Hence ψ is a homeo. □

Ex. $A = k[x_0, \dots, x_n]$, $f = x_i$

$\operatorname{Proj} A \supseteq D(x_i) \underset{\psi}{\cong} \operatorname{Spec} k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ $\forall i$ $\Rightarrow$ $\operatorname{Proj} A = \bigcup_{i=0}^{n} D(x_i)$ covering.

Structure sheaf on Proj

$D(f) \cong \operatorname{Spec}(A_f)_0$ homeo. If we want this to also respect the ring structure, it already gives us what the structure sheaf should be.

$\mathcal{B} := \{ D(f) \mid f \in A \text{ homog.} \}$

$\mathcal{O}^{\mathcal{B}}_{\operatorname{Proj} A}(D(f)) := (A_f)_0$ defines a presheaf on $\mathcal{B}$.

Lemma. If $D(g) \subseteq D(f)$ with g homog. then $\mathcal{O}^{\mathcal{B}}_{\operatorname{Proj} A}(D(g)) = \mathcal{O}^{\mathcal{B}}_{\operatorname{Spec}(A_f)_0}(D(\bar{g}))$ where $\bar{g} = \frac{g^{\deg f}}{f^{\deg g}}$.

Pf: $\text{RHS} = ((A_f)_0)_{\bar{g}} = ((A_f)_g)_0 \underset{\substack{\uparrow \\ g^m = f \cdot a \text{ for some } m \geq 1}}{=} (A_g)_0$ □

Cor. $\psi_* \mathcal{O}^{\mathcal{B}}_{\operatorname{Proj} A}|_{D(f)} = \mathcal{O}_{\operatorname{Spec}(A_f)_0}$ (considered as a sheaf on the basic opens of $\operatorname{Spec}(A_f)_0$) □

Hence $\mathcal{O}^{\mathcal{B}}_{\operatorname{Proj} A}$ is a sheaf on $\mathcal{B}$.

Def. The structure sheaf on Proj A is the sheaf associated to $\mathcal{O}^{\mathcal{B}}_{\operatorname{Proj} A}$.

Lemma. $(\operatorname{Proj} A, \mathcal{O}_{\operatorname{Proj} A})$ is a scheme.

Pf: $(D(f), \mathcal{O}_{\operatorname{Proj} A}|_{D(f)}) \underset{\text{LRS}}{\cong} \operatorname{Spec}(A_f)_0$ □

Def. $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \dots, x_n]$ projective n-space over R.

$\mathbb{P}^n_R = \bigcup_{i=0}^{n} D(x_i)$, $(D(x_i), \mathcal{O}_{\mathbb{P}^n}|_{D(x_i)}) \cong \operatorname{Spec} R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] = \mathbb{A}^n_R$

Thm. $\mathbb{P}^n_R \longrightarrow \operatorname{Spec} R$ is proper. (Use VC)

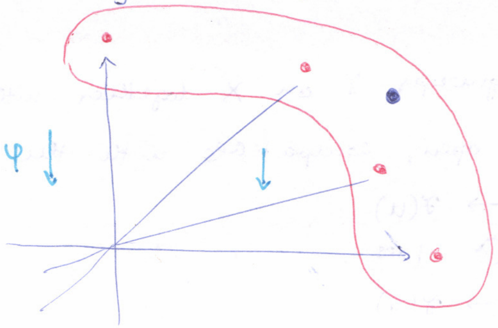
Ex. $I \subseteq A$ homog $\Rightarrow$ $\operatorname{Proj}(A/I) \hookrightarrow \operatorname{Proj} A$ closed immersion

Ex. Warning: not every graded hom. $\varphi: A \to B$ induces a morphism $\operatorname{Proj} B \to \operatorname{Proj} A$..

$$\begin{aligned} \varphi: k[x,y] &\longrightarrow k[x,y] \\ x &\longmapsto x \\ y &\longmapsto x \end{aligned}$$

$\varphi^{-1}((-bx+ay)) = (0)$ if $a \neq 0$

but $\varphi^{-1}((x)) = (x,y) \notin \operatorname{Proj} k[x,y]$



Def. $A = k[x_0, \dots, x_n]$ with $\deg x_i = a_i$. Then $\mathbb{P}^n(a_0, \dots, a_n) := \operatorname{Proj} A$ is the weighted projective space.

Ex. $\mathbb{P}(1,1,n)$: $D(x) \cong \operatorname{Spec}[x, y, z, \frac{1}{x}]_0 = \operatorname{Spec}[\frac{y}{x}, \frac{z}{x^n}] = \mathbb{A}^2_k$, $D(y) \cong \mathbb{A}^2_k$,

$D(z) = \operatorname{Spec} k[x,y,z,\frac{1}{z}]_0$ $\quad x^a y^b z^c \in k[x,y,z,\frac{1}{z}]_0 \Leftrightarrow a,b \geq 0$ & $a+b+nc=0$

$\Rightarrow a+b \equiv 0 \pmod n$. Let $X := \frac{x}{z^{1/n}}$, $Y := \frac{y}{z^{1/n}}$.

$$\Rightarrow \quad k[x,y,z,\tfrac{1}{z}]_0 = k[X^n, X^{n-1}Y, \dots, Y^n]$$

Case $n=2$: $D(z) \cong \operatorname{Spec} k[X^2, XY, Y^2] = \operatorname{Spec} k[x_0,x_1,x_2]/_{x_1^2 = x_0 x_2}$

Case $n=3$:
$k = \mathbb{C}$

$\mathbb{C}[X^n, X^{n-1}Y, \dots, Y^n] = k[X,Y]^{\mathbb{Z}_n}$

$\Rightarrow$ "$\operatorname{Spec}(\dots)$" $\cong$ "$\mathbb{C}/\mathbb{Z}_n$" (not defined)

$\mathbb{Z}_n \curvearrowright k[X,Y]$ by $X \mapsto \xi X$, $Y \mapsto \xi Y$ where $\xi = \exp(2\pi i/n)$

Question. Why is A non-negatively graded in the def of Proj?

Ex. $A = \mathbb{C}[x,y]$, $\deg x = 1$, $\deg y = -1$.

"Proj" $A := \{$homog prime ideals in $\mathbb{C}[x,y]\}$

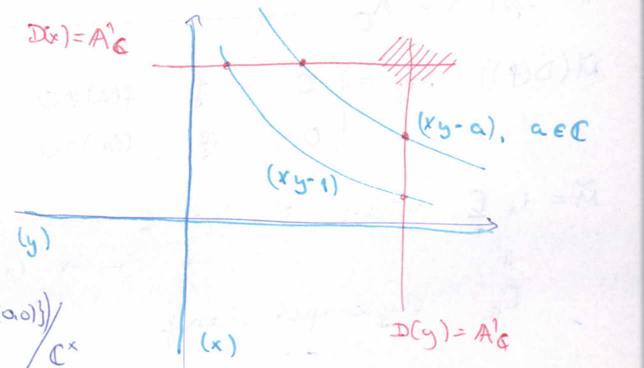
$p \subseteq \mathbb{C}[x,y]$ is a homog prime $\Leftrightarrow$ irred closed subset of $\mathbb{C}^2$, invariant under $t\cdot(x,y) = (tx, \frac{1}{t}y)$ for $t \in \mathbb{C}^\times$

"Proj" $A = \{(0)\} \amalg \{(xy-a) \mid a \in \mathbb{C}\} \amalg \{(x)\} \amalg \{(y)\}$

$D(x) = \operatorname{Spec} k[x,y,\frac{1}{x}]_0 = \operatorname{Spec} k[xy] = \mathbb{A}^1_{\mathbb{C}}$

$D(y) = \operatorname{Spec} k[x,y] = \mathbb{A}^1_{\mathbb{C}}$

$\Rightarrow$ "Proj" $A = \mathbb{A}^1_{\mathbb{C}} \bigcup\limits_{\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}} \mathbb{A}^1_{\mathbb{C}} =$ ——:—— "=" $(\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{C}^\times$



Quasicoherent $\mathcal{O}_X$-modules

07. 01. 2019

<u>Notation</u>. $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$

$\Gamma(\mathcal{F}) := \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$

Let $(X, \mathcal{O}_X)$ be a scheme.

<u>Def.</u> An <u>$\mathcal{O}_X$-module on X</u> is a sheaf of abelian groups $\mathcal{F}$ on X together with an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ for all $U \subseteq X$ open, compatible with the restriction morphisms, i.e. $\forall V \subseteq U$:

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d, "\mathrm{res}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(U) \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

<u>Lemma:</u> The category of $\mathcal{O}_X$-modules is abelian.

"Pf": Most of this is immediate from the properties of sheaves of ab. gps, (which we haven't verified either).

Nts Ker, Coker, $\oplus$ ~~car~~ can be endowed with an $\mathcal{O}_X$-structure. This is done on presheaves first, then one has to sheafify and then diagrams like this commute:

$$\begin{tikzcd} \mathcal{O}_X \times \mathrm{Coker}^{\mathrm{pre}} \arrow[r] \arrow[d, "\mathrm{sh}"] & \mathrm{Coker}^{\mathrm{pre}} \arrow[d, "\mathrm{sh}"] \\ \mathcal{O}_X \times \mathrm{Coker} \arrow[r, "\exists!"] & \mathrm{Coker} \end{tikzcd}$$

$\square$

<u>Key Example</u>. $X := \operatorname{Spec} A$, M an A-module. We define the module $\tilde{M}$.

On basic open subsets, $\tilde{M}(D(f)) := M_f \qquad (f \in A)$

<u>Prop / Def.</u> $\tilde{M}$ is a sheaf on the basic open subsets of X. Then let <u>$\tilde{M}$</u> denote the induced sheaf on X too.

Pf: Basically the same as for $\mathcal{O}_X$, just with modules instead of rings. $\square$

On basic opens, define an $\mathcal{O}_X$-module structure:

$$\begin{matrix} A_f & \times & M_f & \longrightarrow & M_f \\ \shortparallel & & \shortparallel & & \\ \mathcal{O}_X(D(f)) & & \tilde{M}(D(f)) & & \end{matrix}$$

Then endow $\tilde{M}$ with the induced $\mathcal{O}_X$-module structure.

<u>Ex.</u> $A = \mathbb{C}[x]$, $M = \mathbb{C}$ with x acting by multiplication by $a \in \mathbb{C}$

$X = \operatorname{Spec} A = \mathbb{A}^1_{\mathbb{C}}$

$$\tilde{M}(D(f)) = M_f = \begin{cases} \mathbb{C} & \text{if } f(a) \neq 0 \iff (x-a) \in D(f) \\ 0 & \text{if } f(a) = 0 \iff (x-a) \notin D(f) \end{cases}$$

$\tilde{M} = i_* \underline{\mathbb{C}}$ where $i : \operatorname{Spec} \mathbb{C} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$

$\qquad \shortparallel \qquad\qquad\qquad\qquad * \longmapsto (x-a)$

$\mathbb{C}_a$ skyscraper sheaf

Algebraic Geometry I, lecture 21

Ex. $M = A$, $\tilde{M}(D(f)) = A_f = \mathcal{O}_X(D(f)) \Rightarrow \tilde{A} = \mathcal{O}_X$

Thus this tilde construction in fact generalises the structure sheaf construction for affine schemes.

What does $\tilde{}$ do with morphisms?

Let $u: M \to N$ be an A-module homomorphism, and let $u_f : M_f \to N_f$ be the localised homomorphism.

We have
$$\begin{tikzcd} M_f \arrow[r, "u_f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & N_f \arrow[d] \\ M_g \arrow[r, "u_g"'] & N_g \end{tikzcd}$$
whenever $D(g) \subseteq D(f)$, i.e. $g = f^n \cdot a$ where the vertical arrow on the left sends $\frac{m}{f^l} \longrightarrow \frac{m \cdot f^{n-l} \cdot a}{f^l \cdot f^{n-l} \cdot a}$.

Let $\tilde{u} : \tilde{M} \longrightarrow \tilde{N}$ be the induced morphism of $\mathcal{O}_X$-modules. Since the u_f's commute with restrictions, this $\tilde{u}$ is well-defined.

Prop. $\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$

$u \longmapsto \tilde{u}$

with the inverse map being $\Gamma = \Gamma(X, -)$:

for $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$, $\Gamma(X, \varphi) : \underset{\mathcal{F}(X)}{\underset{\parallel}{\Gamma(X, \mathcal{F})}} \longrightarrow \underset{\mathcal{G}(X)}{\underset{\parallel}{\Gamma(X, \mathcal{G})}}$

Pf: $\Gamma \circ \tilde{} = \mathrm{id}$ by construction. ✓

Wts $\tilde{} \circ \Gamma = \mathrm{id}$. Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$, $u := \Gamma(X, \varphi) \in \mathrm{Hom}_A(M, N)$.

Consider $\tilde{u} : \tilde{M} \longrightarrow \tilde{N}$. For all $f \in A$ we have a comm diag of A-modules
$$\begin{tikzcd} M \arrow[r, "u=\Gamma(\varphi)"] \arrow[d, "\mathrm{loc}"'] & N \arrow[d, "\mathrm{loc}"] \\ M_f \arrow[r, shift left, "\varphi_{D(f)}"] \arrow[r, shift right, "\tilde{u}_{D(f)}"'] & N_f \end{tikzcd}$$
for both arrows at the bottom. Hence $\varphi_{D(f)} = \tilde{u}_{D(f)}$ $\forall f$, and thus $\varphi = \tilde{u}$.

Cor. $\tilde{}: A\text{-Mod} \hookrightarrow \mathcal{O}_X\text{-Mod}$, i.e. the category of A-modules is equivalent to the category of $\mathcal{O}_X$-modules of the form $\tilde{M}$. (Recall $X = \mathrm{Spec}\, A$.)

Lemma 1. 1) $M \to N \to P$ is exact $\Leftrightarrow$ $\tilde{M} \to \tilde{N} \to \tilde{P}$ is exact

2) For $u \in \mathrm{Hom}(M, N)$: $\widetilde{\mathrm{Ker}(u)} = \mathrm{Ker}(u)$,

$\widetilde{\mathrm{Im}}(u) = \mathrm{Im}(\tilde{u})$,

$\widetilde{\mathrm{Coker}}(u) = \mathrm{Coker}(\tilde{u})$

3) $\widetilde{\bigoplus_i M_i} = \bigoplus_i \tilde{M}_i$

Pf: 1) $M \to N \to P$ exact $\iff$ $\forall \mathfrak{p} \in \operatorname{Spec} A$: $M_\mathfrak{p} \to N_\mathfrak{p} \to P_\mathfrak{p}$ is exact

$\iff$ $\tilde{M} \to \tilde{N} \to \tilde{P}$ is exact

2) $0 \to \operatorname{Ker}(u) \to M \xrightarrow{u} N \to 0$. Apply $\sim$ and 1):

$0 \to \widetilde{\operatorname{Ker}(u)} \to \tilde{M} \xrightarrow{\tilde{u}} \tilde{N} \to 0$ is exact $\Rightarrow$ $\widetilde{\operatorname{Ker}(u)} = \operatorname{Ker}(\tilde{u})$.

Same for Im and Coker.

3) Exercise. □

Rmk. Using the Cor, the Lemma is automatic: we have an equivalence of cats, they are both additive and one is abelian.

Let $X = \operatorname{Spec} A$.

$$\{\mathcal{O}_X\text{-modules}\} \supsetneq \{\tilde{M} \mid M \text{ is an } A\text{-module}\} \simeq \{A\text{-modules}\}$$

abelian cat

abelian cat

this category is nice

this one is crazy

Ex. The above inclusion is proper: $A := \mathbb{C}[x]$, $X = \mathbb{A}^1_\mathbb{C}$, $\mathcal{F}(U) := \begin{cases} \mathcal{O}_X(U) & \text{if } 0 \notin U \\ 0 & \text{if } 0 \in U \end{cases}$

$\Rightarrow$ $\mathcal{F}$ is $\mathcal{O}_X$ locally away from 0, but $\mathcal{F}$ is 0 in every nbhd of 0.

This $\mathcal{F}$ is an $\mathcal{O}_X$-module, but $\Gamma(X, \mathcal{F}) = 0$,

hence $\mathcal{F} \neq \widetilde{\Gamma(X, \mathcal{F})}$.

Now let X be an arbitrary scheme. The above inclusion will have the following generalisation: $\mathcal{O}_X\text{-Mod} \supsetneq \operatorname{QCoh}(X)$

Thm/Def. X a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. TFAE:

(1) $\forall U \subseteq X$ open affine: $\mathcal{F}|_U \cong \tilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M.

(2) X has an open affine cover $\{U_i\}_i$ s.t. $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$ for $\Gamma(U_i, \mathcal{O}_X)$-modules M_i.

(3) $\forall x \in X$ $\exists x \in U \subseteq X$ open and an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X|_U^{\oplus I} \longrightarrow \mathcal{O}_X|_U^{\oplus J} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$ for some index sets I, J.

(4) $\forall U = \operatorname{Spec} A \subseteq X$ open affine $\forall f \in A$: $\Gamma(U, \mathcal{F})_f \xrightarrow{\operatorname{res}_f} \Gamma(D(f), \mathcal{F})$ is an iso.

If (1—4) hold, we call $\mathcal{F}$ quasicoherent. | Here res_f is the morphism induced by res on the localisation.

(1) → (2) is clear. ✓

(2) → (3): let $A_i := \Gamma(U_i, \mathcal{O}_X) = \mathcal{O}_X(U_i)$.

Choose a resolution $A_i^{\oplus I} \longrightarrow A_i^{\oplus J} \longrightarrow M_i \longrightarrow 0$.

Apply tilde: $\widetilde{A}_i^{\oplus I} \longrightarrow \widetilde{A}_i^{\oplus J} \longrightarrow \widetilde{M}_i \longrightarrow 0$ is still exact

Since $\widetilde{A}_i = \mathcal{O}_X|_{U_i}$ and the U_i cover X, this shows (3). ✓

Rmk. In (3) we may shrink U and the sequence will still be exact.

(3) → (4): Let $U = \operatorname{Spec} A$, $f \in A$. By the Rmk $U = \bigcup_{i=1}^{m} D(g_i)$ finite cover of basic open sets $(g_i \in A)$ s.t. $\mathcal{O}_U|_{D(g_i)}^{\oplus I_i} \xrightarrow{\alpha_i} \mathcal{O}_U|_{D(g_i)}^{\oplus J_i} \longrightarrow \mathcal{F}|_{D(g_i)} \longrightarrow 0$ for suitable I_i, J_i.

$\mathcal{O}_U|_{D(g_i)} = \widetilde{A}_{g_i}$ and $\alpha_i = \widetilde{a}_i$ by Prop for

some $a_i: A_{g_i}^{\oplus I} \longrightarrow A_{g_i}^{\oplus J}$

Hence $\mathcal{F}|_{D(g_i)} = \widetilde{\operatorname{Coker}(a_i)}$ $\Rightarrow$ $\underbrace{\mathcal{F}|_{D(g_i)}(D(\bar{f}))}_{\mathcal{F}(D(g_i) \cap D(f))} = \mathcal{F}(D(g_i))_{\bar{f}}$ (*)

where $\bar{f}$ is the img of f in A_{g_i}.

Applying the same argument for $D(g_i) \cap D(g_j)$ we obtain:

$$\mathcal{F}(D(g_ig_j) \cap D(f)) = \mathcal{F}(D(g_ig_j))_f \qquad (**)$$

We have an exact sequence

$$0 \longrightarrow \mathcal{F}(U)_f \longrightarrow \underbrace{\coprod_{i=1}^{m} \mathcal{F}(D(g_i))_f}_{\cong \ (*)} \longrightarrow \underbrace{\coprod_{i=1}^{m} \coprod_{\substack{j=1 \\ j \neq i}}^{m} \mathcal{F}(D(g_ig_j))_f}_{\cong \ \mathrm{res}_f} \longrightarrow 0.$$

$\big| \mathrm{res}_f$ We may localise.

$\downarrow$

$$0 \longrightarrow \mathcal{F}(D(f)) \longrightarrow \coprod_{i=1}^{m} \overset{\mathrm{res}_f}{\mathcal{F}(D(g_i) \cap D(f))} \longrightarrow \coprod_{i \neq j} \mathcal{F}(D(g_ig_j) \cap D(f))$$

This proves (4). ✓

(4) → (1): $U = \operatorname{Spec} A$, $M = \Gamma(U, \mathcal{F})$.

$\forall f \in A$:

$$\begin{tikzcd} M \arrow[rr, "\mathrm{res}"] \arrow[dr] & & \mathcal{F}(D(f)) \\ & M_f \arrow[ur, "\cong \text{ by } (4)"'] & \end{tikzcd}$$

$\Rightarrow \widetilde{M} \xrightarrow{\sim} \mathcal{F}$ on a basis of basic open subsets on X, hence on X as well. ✓

Cor. 1) Kernels, cokernels, images of morphisms of qcoh sheaves are qcoh.

2) Direct sums of qcoh sheaves are qcoh.

In particular, the full subcategory of qcoh $\mathcal{O}_X$-modules (in $\mathcal{O}_X$-Mod) is abelian. ⧄

Pf: $0 \longrightarrow \operatorname{Ker}(\tilde{\varphi}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \operatorname{Coker}(\varphi) \longrightarrow 0$

Restrict to $U = \operatorname{Spec} A \subseteq X$ open affine.

$\mathcal{F}|_U = \tilde{M}$ where $M = \Gamma(U, \mathcal{F})$

$\mathcal{G}|_U = \tilde{N}$ where $N = \Gamma(U, \mathcal{G})$

$\varphi_U = \tilde{a}$ for some $a: M \to N$, by Prop.

$$\Rightarrow \quad 0 \to \operatorname{Ker}(\varphi)|_U \to \tilde{M} \xrightarrow{\tilde{a}} \tilde{N} \longrightarrow \operatorname{Coker}(\varphi)|_U \to 0$$

$$\Rightarrow \quad \underbrace{\phantom{\operatorname{Ker}(\varphi)|_U}}_{= \widetilde{\operatorname{Ker}(a)}} \quad \text{and} \quad \underbrace{\phantom{\operatorname{Coker}(\varphi)|_U}}_{= \widetilde{\operatorname{Coker}(a)}}$$

□

Ex. X a scheme, $i: Z \hookrightarrow X$ a closed immersion.

Consider the exact sequence $0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Z \underbrace{\longrightarrow 0}_{i \text{ is a closed immersion}}$

where $\mathcal{I}_Z$ is a sheaf of ideals.

Prop. $\mathcal{I} \subseteq \mathcal{O}_X$ any sheaf of ideals. Then $\mathcal{I}$ is qcoh iff $\mathcal{I} = \mathcal{I}_Z$ for some closed subscheme $Z \subseteq X$.

Pf: When X is affine, $X = \operatorname{Spec} A$.

Suppose $Z \subseteq X$ is a closed subscheme, $Z = \operatorname{Spec}(A/I)$

$\forall f \in A$: $(i_* \mathcal{O}_Z)(D(f)) = \mathcal{O}_Z(D(\bar{f})) = \underbrace{(A/I)_{\bar{f}}}_{= \widetilde{A/I}\,(D(f))}$ where $\bar{f}$ is the img of f in A/I

$\Rightarrow i_* \mathcal{O}_Z = \widetilde{A/I}$ on $\operatorname{Spec} A$

$\Rightarrow \tilde{A} = \mathcal{O}_X$, $i_* \mathcal{O}_Z$ are qcoh $\Rightarrow \mathcal{I}_Z$ is qcoh by Lemma 1. ✓

Now let $\mathcal{I}$ be qcoh, i.e. $\mathcal{I} = \tilde{I}$ for some $I \subseteq A$ ideal.

$$\begin{tikzcd} 0 \arrow[r] & \tilde{I} \arrow[r] & \mathcal{O}_X \arrow[r] & \widetilde{A/I} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & & & i_* \mathcal{O}_Z & \end{tikzcd}$$

where $Z := \operatorname{Spec} A/I$ by the discussion above in this proof.

$\Rightarrow \mathcal{I} = \tilde{I} = \mathcal{I}_Z$

□

Constructions on $\mathcal{O}_X$-modules

Tensor product. $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules, then $\mathcal{F} \otimes \mathcal{G}$ is defined by:

$$\left(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}\right) := \left(U \mapsto \underbrace{\left(\mathcal{F}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{G}(U)\right)}_{\mathcal{O}_X(U)\text{-module}}\right)^{sh}$$

If $\mathcal{F}, \mathcal{G}$ are qcoh, so is $\mathcal{F} \otimes \mathcal{G}$.

$$\left(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}\right)(U) = \mathcal{F}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{G}(U) \quad \forall U \subseteq X \text{ open aff}$$

Hom sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

Dual $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$

Pushforward & pullback. $f: X \longrightarrow Y$, $\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on Y

pushforward: $f_* \mathcal{F}$, pullback: $f^* \mathcal{G}$

$\mathcal{G}$ qcoh $\Rightarrow$ $f^*\mathcal{G}$ qcoh

$\mathcal{F}$ qcoh, f qcqs $\Rightarrow$ $f_* \mathcal{F}$ qcoh

Def. An $\mathcal{O}_X$-module is locally free if $\forall x \in X$ $\exists x \in U$ open nbhd and $\exists I$ idx set, $\mathcal{F}|_U = \mathcal{O}_X|_U^{\oplus I}$. Then $\mathrm{rk}_x(\mathcal{F}) := \# I$.

Note that $\mathrm{rk}_{\_}(\mathcal{F})$ is locally constant on X.

1) $\mathcal{F}(x) := \mathcal{F}_x / m_x \mathcal{F}_x$ where $m_x \subset \mathcal{O}_{X,x}$. $\mathcal{F}(x)$ is a vector space $/k(x)$ of dim $= \# I$.

2) $\mathcal{O}_X|_U^{\oplus I} \simeq \mathcal{O}_X|_U^{\oplus I'} \Rightarrow \# I = \# I'$

Rmk. loc free $\Rightarrow$ qcoh. $U = \mathrm{Spec}\, A$, $\mathcal{F}|_U = \mathcal{O}_X|_U^{\oplus I} = \widetilde{A}^{\oplus I}$

Rmk. Later if $\mathrm{rk} < \infty$ and constant $\longrightarrow$ vb

Def. Invertible sheaf: loc free of constant rk 1.

Rmk. If $\mathcal{L}, \mathcal{M}$ are invertible, so is $\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$:

$$\mathcal{L}|_U \simeq \mathcal{O}_X|_U \simeq \mathcal{M}|_U, \quad (\mathcal{L} \otimes \mathcal{M})|_U \simeq \mathcal{O}_X|_U \underset{\mathcal{O}_X|_U}{\otimes} \mathcal{O}_X|_U \simeq \mathcal{O}_X|_U$$

$$\mathcal{L}^\vee \otimes \mathcal{L} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X \quad \text{exc.}$$
$$\mathrm{id} \mapsto 1$$

$\mathrm{Pic}(X) := \{\text{invertible sheaves on } X\}$ with $\otimes$ and $-^\vee$ is the Picard group of X

Rmk. $\mathcal{L}$ inv'ble sheaf, $s \in \Gamma(X, \mathcal{L})$, $x \in X$

we have $\mathcal{L}_x \cap \mathcal{O}_{X,x} \supset m_x$

$s(x) :=$ img of s under
$$\begin{tikzcd} \Gamma(X,\mathcal{L}) \arrow[r] & \mathcal{L}_x \arrow[r] & \mathcal{L}_x / m_x \mathcal{L}_x = \mathcal{L}(x) \\ s \arrow[r, mapsto] & s_x & \end{tikzcd}$$

$D(s) := \{x \in X \mid s(x) \neq 0\}$

Let $x \in U \subseteq X$ be open s.t. $\mathcal{L}|_U \xrightarrow{\varphi} \mathcal{O}_X|_U$.

Let $f_s := \varphi(s|_U) \in \mathcal{O}_X(U)$

$$\begin{tikzcd} s \in \Gamma(U,\mathcal{L}) \arrow[r] \arrow[d, "\varphi_U"', "\cong"] & \mathcal{L}_x \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(x) \ni s(x) \arrow[d, "\varphi"] \\ \mathcal{O}_X(U) \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & k(x) \end{tikzcd}$$

$\Rightarrow f_s(x) = \varphi_x(s(x))$, $\quad D(s) \cap U = D(f_s)$

$V(s) := \{x \in X \mid s(x) = 0\}$ is closed with subscheme structure defined by

$V(s) \cap U = V(f_s) = \operatorname{Spec}(A/f_s)$ where $U = \operatorname{Spec} A$.

(Note: $V(s)$ is an eff Cart. div.)

Coherent sheaves

Def. An A-module M is

- finitely generated if $\exists A^{\oplus r} \twoheadrightarrow M$, $r \geq 0$
- of finite presentation if $\exists A^{\oplus s} \longrightarrow A^{\oplus r} \longrightarrow M \longrightarrow 0$ exact, $r, s \geq 0$
- coherent if fin gen and $\forall \varphi: A^{\oplus r} \longrightarrow M$ morphism: $\operatorname{Ker} \varphi$ is fin gen.

Prop. coherent $\Rightarrow$ fin pres $\Rightarrow$ fin gen, and we have equivalence if A is noeth. ☐

Def. A qcoherent sheaf $\mathcal{F}$ is of finite type / of finite presentation / coherent if

$\forall U \subseteq X$ affine open: $\mathcal{F}(U)$ is oft / ofp / coh.

Prop. X noeth $\Rightarrow$ oft = ofp = coh. ☐

Prop. For any X, $\operatorname{Coh}(X)$ is abelian. ☐

Def. $\operatorname{Supp} \mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\}$

Lemma. $\mathcal{F}$ qcoh oft $\Rightarrow \operatorname{Supp} \mathcal{F}$ is closed.

Pf: U open affine, $\varphi: \mathcal{O}_X(U)^{\oplus r} \twoheadrightarrow \mathcal{F}(U)$. Let $m_i := \varphi(e_i)$ where $e_i = (0, \dots, \overset{i}{1}, \dots, 0)$

$\operatorname{Supp} \mathcal{F} = \{x \in X \mid m_{i,x} \neq 0$ for some $i = 1, \dots, r\} = \bigcup_{i=1}^{r} \operatorname{Supp} m_i = \underbrace{\bigcup_{i=1}^{r}}_{\text{fin}} \underbrace{V(\operatorname{Ann} m_i)}_{\text{closed}}$

$\mathcal{O}_{X,x}^{r} \twoheadrightarrow \mathcal{F}_x$

Quasicoherent sheaves on Proj A

$A = \bigoplus_{d \geq 0} A_d$, $X = \text{Proj } A = \{ A_+ \not\subseteq p \subseteq A$ homog. prime$\}$. $D(f) = \{ p \in \text{Proj } A \mid f \notin A\, p \}$
$\simeq \text{Spec}(A_f)_0$ open subsch.

$M = \bigoplus_{d \geq 0} M_d$ graded A-module $(A_d \cdot M_e \subseteq M_{d+e})$

$\tilde{M}(D(f)) := (M_f)_0$ on basic opens

$D(g) \subset D(f)$ for $f, g \in A_+$ $\Rightarrow g \in \sqrt{(f)}$, $g^n = f \cdot a$, $a \in A$

$$\tilde{M}(D(f)) = (M_f)_0 \longrightarrow (M_g)_0 = \tilde{M}(D(g)) \quad \text{restriction map}$$

$$\frac{m}{f^k} \longmapsto \frac{m \cdot a^k}{f^k \cdot a^k} = \frac{m a^k}{g^{nk}}$$

Lemma. $\tilde{M}$ is a sheaf of the basic opens of Proj A.

Def. $\tilde{M} :=$ the induced sheaf on Proj A with canonical $\mathcal{O}_X$-module structure defined by $(A_f)_0 \times (M_f)_0 \longrightarrow (M_f)_0$

Pf: $D(g) \subset D(f)$, $f, g \in A_+$ $\Rightarrow \tilde{M}(D(g)) = (M_g)_0 \cong \left((M_f)_0\right)_{\bar{g}} = \widetilde{(M_f)_0}\,(D(\bar{g}))$

$$\bar{g} = \frac{g^{\deg f}}{f^{\deg g}}$$

$\Rightarrow \tilde{M}|_{D(f)} = \widetilde{(M_f)_0}$ $\quad (*)$

$\Rightarrow \tilde{M}|_{D(f)}$ is a sheaf on the basic opens of $\text{Spec}(A_f)_0$

Since (basic opens of Proj A) $= \bigcup_{f \in A_+}$ (basic opens of $\text{Spec}(A_f)_0$), this completes the pf.

Lemma. $\tilde{M}$ is qcoh.

Pf: $(*)$ & Thm / Def from lec. 22.

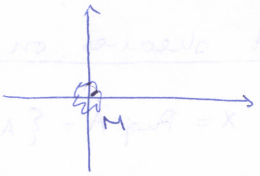
Ex. $M = A$, $\tilde{M} = \mathcal{O}_{\text{Proj } A}$

Ex.^{2} $A = \mathbb{C}[x, y]$, $X = \text{Proj } A = \mathbb{P}^1_{\mathbb{C}}$, $M := \mathbb{C}[x,y] / (ax + by, x^2, xy, y^2)$

$\tilde{M}(D(x)) = (M_x)_0 = 0$ since $M_x = 0$ b/c x acts nilpotently
$\tilde{M}(D(y)) = 0$
$\Big\} \Rightarrow \tilde{M} = 0$.

More gen.: if $\exists N$ s.t. $M_d = 0$ $\forall d \geq N$, then (for any A): $\tilde{M} = 0$. because of the same argument; $\tilde{M}(D(f)) = (M_f)_0$.

Intuition. $k = \mathbb{C}$, $A = \mathbb{C}[x,y]$, "$\mathbb{P}^1_\mathbb{C} = \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^\times$"



"something crazy sits at the origin"

If $\forall d \geq N$, $M_d = 0$ then $m^N \cdot M = 0$, $m = (x,y)$

$\Rightarrow$ M supported at $(0,0)$

Before quotienting we take out $\{0\}$, so we lose info about M.

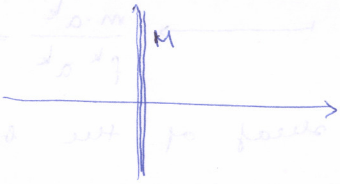
Intuition. $\text{Gr } A\text{-Mod} \xrightarrow{\sim} A\text{-Mod with } \mathbb{C}^\times\text{-action}$

$$(M = \bigoplus M_d) \longmapsto (M, t \in \mathbb{C} \text{ acts on } M_d \text{ by } t^d)$$

M_0: invariants of the $\mathbb{C}^\times$-action

$(M_f)_0$: " " " " on $D(f)$

Ex. 2. (continued) $M := \mathbb{C}[y] = \mathbb{C}[x,y]/(x)$



$\tilde{M}(D(x)) = (M_x)_0 = 0$

$\tilde{M}(D(y)) = (\mathbb{C}[y]_y)_0 = \mathbb{C}$ $\Rightarrow$ M is the skyscraper sheaf on $\mathbb{P}^1_\mathbb{C}$ at $(x,y) = [0,1]$

Lemma. $M \longrightarrow \tilde{M}$, $(u : M \to N) \longmapsto (\tilde{u} : \tilde{M} \to \tilde{N})$ where $\tilde{u}$ is induced by $(u_f)_0 : (M_f)_0 \longrightarrow (N_f)_0$. This defines an exact functor $(\text{Gr } A\text{-Mod}) \longrightarrow (\text{QCoh}(\text{Proj } A))$

Pf: $M \longrightarrow N \longrightarrow P$ exact

$\Rightarrow M_f \to N_f \to P_f$ "

$\Rightarrow (M_f)_0 \to (N_f)_0 \to (P_f)_0$ "

$\Rightarrow \widetilde{(M_f)_0} \to \widetilde{(N_f)_0} \to \widetilde{(P_f)_0}$ " since on affines, $\sim$ is exact

$\Rightarrow \tilde{M} \to \tilde{N} \to \tilde{P}$ "

Ex. $I = (x)$ $\Rightarrow$ $0 \to I \to \mathbb{C}[x,y] \to \mathbb{C}[y] \to 0$

$$\Rightarrow 0 \to \underbrace{\tilde{I}}_{\sim} \to \mathcal{O}_X \to \mathbb{C}_{[0,1]} \to 0$$

inv'ble sheaf of ideals, $\simeq \mathcal{O}_X(-1)$, see below

Serre twisting sheaves

$X = \text{Proj } A$, M gr mod, $n \in \mathbb{Z}$

n-twisted module: $M(n) = \bigoplus_{d \in \mathbb{Z}} M(n)_d$ where $M(n)_d = M_{n+d}$

Ex. $A = \mathbb{C}[x,y]$, $A(2)_{-3} = 0$, $A(2)_{-2} = \mathbb{C}$, $A(2)_{-1} = \mathbb{C}[x,y]$

$M = (x)$ $M_0 = 0 = A_{-1}$, $M_1 = \mathbb{C} = A_0$, ... $\Big\}$ $M \simeq A(-1)$

Def. $\mathcal{O}_X(n) := \widetilde{A(n)}$

Algebraic Geometry 1, lecture 22–23

On D(f): $\mathcal{O}_X(n)(D(f)) = (A(n)_f)_0 = (A_f)_n$

If $\deg f \mid n$: $(A_f)_n \longrightarrow (A_f)_0$

$$\frac{a}{f^k} \longmapsto \frac{a}{f^k \, f^{\frac{n}{\deg f}}}$$

$\Rightarrow \mathcal{O}_X(n)|_{D(f)} = \widetilde{(A_f)_n} \simeq (\widetilde{A_f})_n = \mathcal{O}_X|_{D(f)}$, so $\mathcal{O}_X(n)|_{D(f)} \simeq \mathcal{O}_X|_{D(f)}$ (*)

Prop. A gr rg gen by A_1 over A_0.

a) $\forall n$: $\mathcal{O}_X(n)$ inv'ble sheaf.

b) $\forall M$ gr A-mod: $\widetilde{M(n)} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

In particular, $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$

c) $\mathcal{O}_X(n)^\vee = \mathcal{O}_X(-n)$

Rmk. For an $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. Then $\widetilde{M}(n) = \widetilde{M(n)}$ by b).

Rmk. $\mathbb{Z} \xrightarrow{\varphi} \operatorname{Pic} X$, $n \longmapsto \mathcal{O}_X(n)$. Later: φ iso. for $X = \mathbb{P}^n_k$.

Pf: a) $\operatorname{Proj} A = \bigcup_{f \in A_1} D(f)$ since A is gen by A_1 (The complement is: $\left(\bigcup D(f)\right)^c = V(f \mid f \in A_1) = \{ p \mid A_1 \subseteq p \} = \{ p \mid A_+ \subseteq p \} = \emptyset$)

(*) $\Rightarrow \mathcal{O}_X(n)|_{D(f)} \simeq \mathcal{O}_X|_{D(f)} \Rightarrow \mathcal{O}_X(n)$ loc free of rk 1. ✓

14.01.2019

Errata. 1) $\operatorname{Pic}(X) =$ gp of inv'ble sheaves up to iso

2) A locally free sheaf of rk $r < \infty$ need not be coherent (if X is not noetherian)

An A-mod M is coherent $\Leftrightarrow$ M fin gen & $\forall \varphi: A^r \longrightarrow M$: Ker φ is fin gen

Ex. $k[x_0, x_1, x_2, \dots] / (x_i x_j,\ i,j \geq 1) =: A$, $M := A$ free

$\varphi: A \longrightarrow M$, $p \longmapsto x_1 \cdot p$ Ker $\varphi = \bigoplus_{i=1}^{\infty} k x_i$ is not fin gen

---

We resume the PROOF:

b) We show $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = \widetilde{M \otimes_A N}$ for any graded A-modules M, N

$\forall f \in A_+$ homogeneous: $\left(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}\right)(D(f)) = \widetilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \widetilde{N}(D(f))$ by PSet 8

$= (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$ by def

$$\begin{tikzcd} (M_f)_0 \otimes_{(A_f)_0} (N_f)_0 \arrow[d, "(*)"] & \frac{m}{f^k} \otimes \frac{n}{f^l} \arrow[d, mapsto] \\ \left(\widetilde{M \otimes_A N}\right)(D(f)) = \left((M \otimes_A N)_f\right)_0 & \frac{mn}{f^{k+l}} \end{tikzcd}$$

This map (*) induces a morphism of sheaves

$$\varphi : \widetilde{M} \underset{\mathcal{O}_X}{\otimes} \widetilde{N} \longrightarrow \widetilde{M \underset{A}{\otimes} N}$$

If $\deg f = 1$ then $\varphi|_{D(f)}$ is an iso with the inverse being

$$\frac{m}{f^{\deg m}} \otimes \frac{n}{f^{\deg n}} \longmapsfrom \frac{mn}{f^c}$$

$\operatorname{Proj} A = \bigcup_{f \in A} D(f) \Rightarrow \varphi$ is an iso $\Rightarrow$ (*) is an iso. ✓

c) $\mathcal{O}_X(n) \otimes \mathcal{O}_X(-n) = \mathcal{O}_X$ by b)

$$\left.\begin{array}{l} \underbrace{\mathcal{O}_X(n) \otimes \mathcal{O}_X(n)^\vee}_{\mathcal{O}_X} \otimes \mathcal{O}_X(-n) \simeq \mathcal{O}_X \otimes \mathcal{O}_X(-n) = \mathcal{O}_X(-n) \end{array}\right\} \Rightarrow \mathcal{O}_X(n)^\vee = \mathcal{O}_X(-n)$$

□

$\widetilde{\ } : \operatorname{Gr} A\text{-Mod} \longrightarrow \operatorname{QCoh}(\operatorname{Proj} A)$

Assumption $(\dagger)$: A is gen'd by fin many elts of A_1 over A_0

Given an $\mathcal{O}_X$-module $\mathcal{F}$ on $\operatorname{Proj} A$

$$\underline{\Gamma_*(\mathcal{F})} = \bigoplus_{n \in \mathbb{Z}} \Gamma(\operatorname{Proj} A, \mathcal{F}(n)), \quad \mathcal{F}(n) = \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(n)$$

$$\varphi : \mathcal{F} \to \mathcal{G}, \quad \underline{\Gamma_*(\varphi)} = \bigoplus_{n \in \mathbb{Z}} \varphi(n)$$

$$\Gamma(\varphi(n)) : \Gamma(\operatorname{Proj} A, \mathcal{F}(n)) \longrightarrow \Gamma(\operatorname{Proj} A, \mathcal{G}(n))$$

Given $\varphi : \mathcal{F} \to \mathcal{G}$, we get $\varphi(n) : \mathcal{F}(n) \xrightarrow{\varphi \otimes \mathrm{id}_{\mathcal{O}_X(n)}} \mathcal{G}(n)$

$$\Rightarrow \Gamma(\varphi(n)) : \Gamma(\mathcal{F}(n)) \longrightarrow \Gamma(\mathcal{G}(n)),$$

$$\Rightarrow \Gamma_*(\varphi) := \bigoplus_n \Gamma(\varphi(n))$$

Thm. Under $(\dagger)$, for every qcoh $\mathcal{F}$ on $\operatorname{Proj} A$ there is a natural iso

$$\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

Cor. Under $(\dagger)$, every $\mathcal{F} \in \operatorname{QCoh}(\operatorname{Proj} A)$ is of the form $\widetilde{M}$ for some fin.gen A-module M.

Pf: Let $f \in A$ homog.

$$\begin{tikzcd}
\widetilde{\Gamma_*(\mathcal{F})}(D(f)) \arrow[rrrr] \arrow[d, phantom, "\|"] & & & & \Gamma(D(f), \mathcal{F}) \\
(f_*(\mathcal{F})_f)_0 \arrow[d, bend right] & \frac{m}{f^k} \arrow[rr, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & m|_{D(f)} \otimes f^k \arrow[r, mapsto, "(*)"] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & f^{-k} \cdot m \\
= \left\{ \frac{m}{f^k} \,\middle|\, m \in \Gamma_*(\mathcal{F})_n = \Gamma(\mathcal{F}(n)),\ n = \deg(f) \cdot k \right\} & \frac{\Gamma(X, \mathcal{F}(n))}{f^k} & & \Gamma(D(f), \mathcal{F}(n) \otimes \mathcal{O}_X(-n)) &
\end{tikzcd}$$

$$f^k \in \Gamma(D(f), \mathcal{O}_X(-n)) = (A_f)_{-n}$$

(*) is induced by $\mathcal{O}_X(n) \otimes \mathcal{O}_X(-n) \to \mathcal{O}_X$

Algebraic Geometry I, lecture 23

This induces a morphism $\Gamma_*(\mathcal{F}) \xrightarrow{\beta} \mathcal{F}$.

By the lemma, $\mathcal{F}$ qcoh $\Rightarrow$ β iso. ($1) \Rightarrow \beta$ inj, $2) \Rightarrow \beta$ surj.)

Lemma. X qcqs scheme, $\mathcal{L}$ inv'ble sheaf, $s \in \Gamma(X, \mathcal{L})$, $D(s) = \{ x \in X \mid s(x) \neq 0 \in \mathcal{L}_x / m_x \mathcal{L}_x \}$, $\mathcal{F}$ qcoh $\mathcal{O}_X$-module.

1) Let $t \in \Gamma(X, \mathcal{F})$ s.t. $t|_{D(s)} = 0$. Then $\exists n > 0$ s.t. $t \otimes s^n = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$

2) Let $t' \in \Gamma(D(f), \mathcal{F})$. Then $\exists n > 0$, $\exists t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ s.t. $t|_{D(s)} = t' \otimes s^n$.

Rmk. X qcqs $\Rightarrow$ X covered by fin many open affines, and the intersection of any two such sets is covered by fin many open affines.

Pf OF LEMMA: $X = \bigcup_{i=1}^n U_i$, U_i open aff s.t. $\mathcal{L}|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}_X$, $\varphi_i(s|_{U_i}) =: s_i$

$\mathcal{F}$ qcoh $\Rightarrow$ $\mathcal{F}(U_i)_{s_i} \xrightarrow[\mathrm{res}_{s_i}]{\sim} \mathcal{F}(D(s_i)) = \mathcal{F}(D(s) \cap U_i)$ $(*)$

1) Let $t_i := t|_{U_i}$. By assumption, $t_i|_{U_i \cap D(s)} = 0$.

By $(*)$: $t_i = 0$ in $\mathcal{F}(U_i)_{s_i}$

$\Rightarrow t_i \cdot s_i^{n_i} = 0$ for some $n_i \geq 1$

$\Rightarrow t \otimes s^n = 0$ for $n = \max_i n_i$ (we use qc here). Note that 1) holds w/o qs.

2) By $(*)$: $\exists !\, t_i \in \mathcal{F}(U_i)$ s.t. $t_i|_{D(s) \cap U_i} = t'|_{D(s) \cap U_i}\, s_i^{m_i}$ for some $m_i \geq 1$

Multiply both sides by a power of s_i $\Rightarrow$ wma $m_i = m$ $\forall i$. for some $m \geq 1$

$\Rightarrow t_i|_{D(s) \cap U_i \cap U_j} = t_j|_{D(s) \cap U_i \cap U_j}$

we use qs here: the intersection is qc

1) applied to $(U_i \cap U_j,\ U_i \cap U_j \cap D(s))$ $\Rightarrow$ $\exists N \geq 1$: $t_i \otimes s^N|_{U_i \cap U_j} = t_j \otimes s^N|_{U_i \cap U_j}$

Glueing $\Rightarrow$ $\exists t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{m+N})$ s.t. $t|_{U_i} = t_i \otimes s^N$

$\Rightarrow t|_{U_i \cap D(s)} = t_i \otimes s^N|_{U_i \cap D(s)} = t' \otimes s^{m+N}$ $\Rightarrow$ $t|_{D(s)} = t' \otimes s^{m+N}$

Application. $A = k[x_0, \dots, x_n]$, $\deg x_i = 1$, $\mathrm{Proj}\, A = \mathbb{P}^n_k$

Every closed subscheme $Z \subseteq \mathbb{P}^n_k$ is of the form $\mathrm{Proj}(A/I) \subseteq \mathbb{P}^n_k$ for some homogeneous ideal $I \subseteq A$.

Fact 1. $\Gamma(\mathbb{P}^n_k, \mathcal{O}(1)) = k[x_0, \dots, x_n]_d$

Fact 2. $\Gamma(X, -)$ is left exact

} exercise

E OF APPL: $Z \subseteq \mathbb{P}^n_k$ closed

$$\begin{tikzcd}
0 \arrow[r] & \underset{\text{qcoh}}{\mathcal{I}_Z} \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_Z \arrow[r] & 0 \\
0 \arrow[r] & \Gamma_*(\mathcal{I}_Z) \arrow[r] \arrow[d, phantom, "\parallel"] & \Gamma_*(\mathcal{O}_X) \arrow[r] \arrow[d, phantom, "\parallel"] & \Gamma_*(\mathcal{O}_Z) & \\
0 \arrow[r] & \underset{\text{ideal}}{I} \arrow[r, phantom, "\subseteq"] & \underset{\text{Fact 1.}}{A} \arrow[r] & A/I \arrow[r] \arrow[u, dashed] & 0 \\
0 \arrow[r] & \tilde{I} \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{O}_X \arrow[r] \arrow[dd, equal] & \widetilde{A/I} \arrow[r] \arrow[dd, dashed, "\cong"] & 0 \\
 & \widetilde{\Gamma_*(\mathcal{I}_Z)} \arrow[d, phantom, "\cong"] & & & \\
0 \arrow[r] & \mathcal{I}_Z \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

$\Gamma(X, -)$ left exact (Fact 2)

$\tilde{\ }$ is exact

$$\Rightarrow \widetilde{A/I} \cong i_* \mathcal{O}_X \Rightarrow V(I) \underset{Set}{=} Z \Rightarrow Z \cong \operatorname{Proj}(A/I)$$ □

Locally free sheaves & vector bundles

Def. Vector bundle of rank r on X: a morphism of schemes $\pi: E \longrightarrow X$

s.t. there is an open cover $X = \bigcup_{i \in I} U_i$ and $\exists \varphi_i$ isos,

$$\begin{tikzcd}
E|_{U_i} = \pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr] & & U_i \times \mathbb{A}^r \arrow[dl] \\
 & U_i &
\end{tikzcd}$$

s.t. the transition functions φ_{ji} are linear over $U_{ij} = U_i \cap U_j$

$$\begin{tikzcd}
U_{ij} \times \mathbb{A}^r & E|_{U_{ij}} \arrow[l, "\varphi_j|_{U_i}"'] \arrow[r, "\varphi_i|_{U_j}"] & U_{ij} \times \mathbb{A}^r \arrow[ll, bend left, "\varphi_{ji}"]
\end{tikzcd}$$

Here φ_{ji} is called linear over U_{ij} if $\forall$ open affine $V = \operatorname{Spec} A \subseteq U_{ij}$

the map $\varphi_{ji}|_V : V \times \mathbb{A}^r \longrightarrow V \times \mathbb{A}^r$ is induced by

$$\begin{aligned} A[x_1, \dots, x_r] &\longrightarrow A[x_1, \dots, x_r] \quad \text{for some } a_{k\ell} \in A. \\ x_k &\longmapsto \sum_{\ell} a_{k\ell} x_k \end{aligned}$$

Rmk. $a_{k\ell} = f_{k\ell}|_V$ for some $f_{k\ell} \in \mathcal{O}_X(U_{ij})$

We say φ_{ji} is given by $\Phi_{ji} = (f_{k\ell})_{k,\ell} \in GL_r(\mathcal{O}_X(U_{ij}))$

Rmk. We have $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$, $\varphi_{ij} = \varphi_{ji}^{-1}$ $\forall i,j,k$. (*)

Conversely, given transition functions φ_i with these properties,

we obtain a vb

$$E := \coprod_{i \in I} (U_i \times \mathbb{A}^r) \Big/ \sim \varphi_{ij}$$

Similarly, let $\mathcal{E}$ be a loc free sheaf of rk r.

$\to \exists$ cover $X = \bigcup U_i$, $\varphi_i : \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus r}$,

$$\begin{tikzcd} \mathcal{O}_{U_{ij}}^{\oplus r} & \mathcal{E}|_{U_{ij}} \arrow[l, "\varphi_j", "\sim"'] \arrow[r, "\varphi_i", "\sim"'] & \mathcal{O}_{U_{ij}}^{\oplus r} \arrow[ll, bend left] \end{tikzcd}$$

φ_{ji} is $\mathcal{O}_X$-linear, so $\mathcal{O}_{U_{ij}}^{\oplus r} \xrightarrow{\varphi_{ji}} \mathcal{O}_{U_{ij}}^{\oplus r}$ $\qquad \varphi_{ji} = \varphi_j \circ \varphi_i^{-1}$

$$e_k \longmapsto \sum_l b_{kl} e_l \qquad b_{kl} \in \mathcal{O}_X(U_{ij})$$

$B_{ij} = (b_{kl}) \in GL_r(\mathcal{O}_X(U_{ij}))$ determines φ_{ji}

As before, we can recover $\mathcal{E}$ from the φ_{ji}.

$$\left\{ \begin{array}{c} \text{vb of rk } r \text{ over } X \\ \text{up to iso} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{transition functions} \\ \text{satisfying } (*)\text{, linear} \\ \text{up to iso} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{loc free sheaves of rk } r \\ \text{on } X \text{ up to iso} \end{array} \right\}$$

$$\begin{array}{c} E \\ \downarrow \pi \\ X \end{array} \longleftrightarrow \mathcal{E} \text{ is the sheaf assoc to } U \mapsto \underbrace{\mathrm{Hom}_U(U, E)^\vee}_{\mathcal{O}_X(U)\text{-module}}$$

The fact that this is an iso will be shown in lecture 24.

<u>Ex.</u> $\mathbb{P}^1 = \mathrm{Proj}\, A$, $A = \mathbb{C}[x,y]$, $\mathcal{O}_X(n) = \widetilde{A(n)}$, $U_0 = D(x)$, $U_1 = D(y)$

$$\mathcal{O}_X(n)(D(x)) = \mathbb{C}[x, \tfrac{1}{x}, y]_n = x^n\, \mathbb{C}[\tfrac{y}{x}] \xrightarrow{\cdot 1/x^n} \mathbb{C}[\tfrac{y}{x}] = (A_x)_0 = \mathcal{O}_X(D(x))$$

$$\mathcal{O}_X(n)(D(y)) = y^n\, \mathbb{C}[\tfrac{x}{y}] \xrightarrow{\cdot 1/y^n} \mathbb{C}[\tfrac{x}{y}] = \mathcal{O}_X(D(y))$$

On $D(x) \cap D(y)$, let $Y := \frac{y}{x}$

$$\begin{tikzcd} \mathbb{C}[Y]_Y & \mathcal{O}_X(n)(D(xy)) \arrow[l, "\varphi_1 = \left(\cdot \frac{1}{y^n}\right)"'] \arrow[r, "\varphi_0 = \left(\cdot \frac{1}{x^n}\right)"'] & \mathbb{C}[Y]_Y = \mathbb{C}[Y, Y^{-1}] \arrow[ll, bend left, "\varphi_{10}"] \end{tikzcd}$$

$\mathcal{O}_X(n)(D(xy)) = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]_n$

$$\varphi_{10}(a) = \frac{x^n}{y^n} \cdot a = \left(\frac{x}{y}\right)^n \cdot a \Rightarrow \varphi_{10}^{\mathcal{O}_X(n)} = \text{multiplication by } \left(\frac{x}{y}\right)^n$$

17.01.2019

Relative Spec construction

$\mathcal{B}$ qcoh $\mathcal{O}_X$-algebra, i.e. a qcoh $\mathcal{O}_X$-module s.t. $\mathcal{B}(U)$ is endowed with an $\mathcal{O}_X(U)$-module structure $\forall U \subseteq X$ open, compatible with restrictions.

Prop. $\exists$ Spec $\mathcal{B}$ scheme and $\pi : \mathrm{Spec}\,\mathcal{B} \longrightarrow X$ s.t. $\forall f : T \to X$ there are bijections

$$\mathrm{Hom}_X(T, \mathrm{Spec}\,\mathcal{B}) = \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{B}, f_*\mathcal{O}_T)$$

functorial in T.

This is a representability statement: it says that Spec $\mathcal{B}$ represents the functor $T/X \longrightarrow \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{B}, f_*\mathcal{O}_T)$

Lemma. $X = \mathrm{Spec}\,A$, $\mathcal{F}$ $\mathcal{O}_X$-mod., M A-mod. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \xrightarrow{\Gamma(\mathrm{Spec}\,A,-)} \mathrm{Hom}_{A\text{-mod}}(M, \mathcal{F}(X))$$ is an isomorphism.

Pf: Inverse given for $u \in \mathrm{Hom}(M, \mathcal{F}(X))$ by $u_f : M_f \xrightarrow{(\mathrm{res}_{D(f)} \circ u)_f} \mathcal{F}(D(f))$, these glue together to $\tilde{u}$.

To show that this is an inverse, consider

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[d] \\ M_f \arrow[r, "\exists!"] & \mathcal{F}(D(f)) \end{tikzcd}$$

(This proof is essentially identical to the one given for qcoh modules.) □

Pf OF PROP: Affine case. $X = \mathrm{Spec}\,A$, $B := \mathcal{B}(X)$ A-algebra with $\tilde{B} = \mathcal{B}$

Then define $\mathrm{Spec}\,\mathcal{B} := \mathrm{Spec}\,B$, and $\pi : \underset{\mathrm{Spec}\,B}{\overset{\Vert}{\mathrm{Spec}\,\mathcal{B}}} \longrightarrow \mathrm{Spec}\,A$ to be the morphism induced by $A \longrightarrow B$.

Here $\mathrm{Hom}_X(T, \mathrm{Spec}\,B) = \mathrm{Hom}_{A\text{-alg}}(B, \underbrace{\mathcal{O}_T(T)}) \longrightarrow = (f_*\mathcal{O}_T)(X)$

$= \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\tilde{B}, f_*\mathcal{O}_T)$ by the Lemma

General case: define $Y = \mathrm{Spec}\,\mathcal{B}$ by gluing.

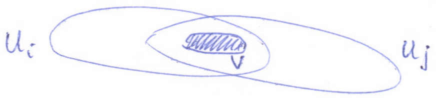
$X = \bigcup_i U_i$, $U_i = \mathrm{Spec}\,A_i$ open affines, $\varphi_i : \mathcal{B}|_{U_i} \xrightarrow{\sim} \tilde{B}_i$ for the A_i-alg $B_i := \mathcal{B}(U_i)$

$Y_i = \mathrm{Spec}\,B_i$

$\pi_i \downarrow$

U_i

Gluing data: $V = \mathrm{Spec}(A') \subseteq U_i \cap U_j$, $V = D(a) \subseteq U_i$, $V = D(b) \subseteq U_j$, $a \in A_i$, $b \in A_j$



$(*)$: $(B_i)_a \simeq \mathcal{B}(V) \simeq (B_j)_b$

$\Rightarrow Y_i|_V = \pi_i^{-1}(V) \xrightarrow{\sim} \pi_j^{-1}(V) = Y_j|_V$

The iso $(*)$ is equal to $\varphi_j|_V \circ \varphi_i|_V^{-1}$, hence satisfies the cocycle condition

The V's above cover $U_i \cap U_j$

$\Rightarrow$ We get an iso $\psi_{ji} : Y_i|_{U_i \cap U_j} \longrightarrow Y_j|_{U_i \cap U_j}$. Let $Y := \left(\coprod Y_i\right)/\sim_{\psi_{ji}}$

Note that here the cocycle condition holds for the ψ_{ji} by the same argument.

Let $f: T \to X$, $\quad T_i := f^{-1}(U_i)$.

$$\mathrm{Hom}_X(T, \mathrm{Spec}\,\mathcal{B}) = \left\{ \begin{tikzcd} T_i \arrow[r, "h_i"] \arrow[rd, "f|_{T_i}"'] & Y_i \arrow[d] \\ & U_i \end{tikzcd} \text{ s.t. } h_i|_{T_i \cap T_j} = h_j|_{T_i \cap T_j} \quad \in \mathrm{Hom}_{U_i}(T_i, \mathrm{Spec}\,\mathcal{B}|_{U_i}) \right\}$$

$$= \left\{ \tilde{h}_i \in \mathrm{Hom}_{\mathcal{O}_{U_i}\text{-alg}}(\tilde{\mathcal{B}}_i, (f|_{T_i})_* \mathcal{O}_{T_i}) \;\middle|\; \tilde{h}_i|_{U_i \cap U_j} = \tilde{h}_j|_{U_i \cap U_j} \right\}$$

$$= \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{B}, f_*\mathcal{O}_T) \qquad \text{since } \mathcal{H}om_{\mathcal{O}_X\text{-alg}}(\mathcal{B}, f_*\mathcal{O}_T) \text{ is a sheaf}$$

Rmk.^{1} Let $\pi: \mathrm{Spec}\,\mathcal{B} \to X$.

Then $\pi_* \mathcal{O}_{\mathrm{Spec}\,\mathcal{B}}(U) \simeq \mathcal{O}_{\mathrm{Spec}\,\mathcal{B}}(\pi^{-1}(U)) = \mathcal{O}_{\mathrm{Spec}\,\mathcal{B}(U)}(\mathrm{Spec}\,\mathcal{B}(U)) = \mathcal{B}(U)$

since $\mathrm{Spec}\,\mathcal{B}|_U = \mathrm{Spec}\,\mathcal{B}(U)$.

$\Rightarrow \pi_* \mathcal{O}_{\mathrm{Spec}\,\mathcal{B}} \simeq \mathcal{B}$.

Rmk.^{2} $\mathrm{Hom}_X(\mathrm{Spec}\,\mathcal{B}, \mathrm{Spec}\,\mathcal{B}') \simeq \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{B}', \pi_* \mathcal{O}_{\mathrm{Spec}\,\mathcal{B}}) = \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{B}, \mathcal{B}')$

Given $\varphi: \mathcal{B} \to \mathcal{B}'$ morphism of $\mathcal{O}_X$-algebras, we get an induced morphism of schemes $\mathrm{Spec}\,\varphi: \mathrm{Spec}\,\mathcal{B}' \to \mathrm{Spec}\,\mathcal{B}$.

Def. A morphism $f: X \to Y$ is affine if $\forall U \subseteq Y$ open affine: $f^{-1}(U)$ is affine.

Upgraded Prop. The functor $\mathcal{B} \mapsto \mathrm{Spec}\,\mathcal{B}$ defines a contravariant equivalence of categories between cat of qcoh $\mathcal{O}_X$-algebras

and cat of X-schemes $\pi: E \to X$ s.t. π is affine.

Pf: Let $\Phi: (\pi: E \to X) \longmapsto (\pi_* \mathcal{O}_E)$.

Rmk 1 $\Rightarrow$ $\Phi \circ \mathrm{Spec} \sim \mathrm{id}$

Conversely: $\mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\pi_* \mathcal{O}_E, \pi_* \mathcal{O}_E) \simeq \mathrm{Hom}(E, \mathrm{Spec}\,\pi_* \mathcal{O}_E)$

$\mathrm{id} \mapsto (\varphi: E \to \mathrm{Spec}\,\pi_* \mathcal{O}_E)$

Over $U = \mathrm{Spec}\,A$, $\varphi|_U \simeq \mathrm{id}$ $\Rightarrow$ φ is an iso.

This is a relative version of the correspondence b/w rings and affine schemes.

Locally free sheaves and vector bundles II

$\mathcal{E}$ loc free sheaf of rk r

$\mathbb{V}(\mathcal{E}) := \operatorname{Spec}(\operatorname{Sym}^\bullet(\mathcal{E}^\vee))$ vector bundle associated to $\mathcal{E}$

not everyone puts a dual here (it doesn't really matter)

$\operatorname{Sym}^\bullet \mathcal{E}$:= sheaf assoc to $U \mapsto \operatorname{Sym}^\bullet_{\mathcal{O}_X(U)} \mathcal{E}(U)$

$$= \bigoplus_{k \geq 0} \mathcal{E}^{\otimes k} / \mathcal{J}$$

where $\mathcal{J} = \langle a \otimes b - b \otimes a \mid a, b \rangle$ is the symmetric ideal

For U open affine: $\operatorname{Sym}^\bullet(\mathcal{E})(U) = \operatorname{Sym}^\bullet_{\mathcal{O}_X(U)} \mathcal{E}(U)$ since $\mathcal{E}$ is qcoh.

Lemma. $\mathbb{V}(\mathcal{E})$ is a vector bundle of rank r.

Pf: Let $X = \bigcup_i U_i$, $U_i = \operatorname{Spec} A_i$ open, $\varphi_i : \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus r}$ trivialisation

We get: $\operatorname{Sym}^\bullet(\varphi_i^\vee) : \operatorname{Sym}^\bullet \mathcal{O}_{U_i}^{\oplus r} = \mathcal{O}_{U_i}[x_1, \dots, x_r] \xrightarrow{\sim} \operatorname{Sym}^\bullet \mathcal{E}^\vee|_{U_i}$

$$\mathcal{O}_{U_i}[x_1, \dots, x_r] = (A[x_1, \dots, x_r])^\sim$$

Apply Spec:

$\psi_i := \operatorname{Spec}(\operatorname{Sym}^\bullet(\varphi_i^\vee)) : \mathbb{V}(\mathcal{E})|_{U_i} \longrightarrow U_i \times \mathbb{A}^r = \operatorname{Spec} \mathcal{O}_{U_i}[x_1, \dots, x_r]$

$$= \operatorname{Spec} A[x_1, \dots, x_r]$$

On overlaps:

$$\begin{tikzcd}
U_{ij} \times \mathbb{A}^r & \mathbb{V}(\mathcal{E})|_{U_{ij}} \arrow[l, "\psi_j|_{U_{ij}}"'] \arrow[r, "\psi_i|_{U_{ij}}"] & U_{ij} \times \mathbb{A}^r \arrow[ll, bend left]
\end{tikzcd}$$

$$\psi_{ji}|_{..} = \psi_j|_{..} \circ \psi_i|_{..}$$

$$\psi_{ji}^\# = (\psi_i^{-1})^\# \circ (\psi_j)^\#$$

$$= \operatorname{Sym}^\bullet(\varphi_i^\vee)^{-1} \circ \operatorname{Sym}^\bullet(\varphi_j^\vee)$$

$$= \operatorname{Sym}((\varphi_i^{-1})^\vee \circ \varphi_j^\vee)$$

$$= \operatorname{Sym}((\varphi_j \circ \varphi_i^{-1})^\vee)$$ linear on U_{ij}. ☐

Upgraded Lemma. $\mathbb{V} : \mathcal{E} \longmapsto \mathbb{V}(\mathcal{E})$

$(\mathcal{E} \xrightarrow{\varphi} \mathcal{F}) \longmapsto (\mathbb{V}(\mathcal{E}) \longrightarrow \mathbb{V}(\mathcal{F}))$ defines a covariant equivalence of categories between loc free sheaves of rk r on X and vector bundles of rk r on X.

The inverse functor is taking sections:

$(\pi : E \longrightarrow X) \longmapsto (\operatorname{Hom}_X(-, \mathcal{F}) : U \longmapsto \operatorname{Hom}_X(U, E) = \{ s : U \to E \mid \pi \circ s = \mathrm{id}_U \})$

Pf: By the Upgraded Prop: $\mathbb{V}$ is an equivalence with inverse $E \mapsto \left[\pi_* \mathcal{O}_E\right]^\vee_{\deg 1}$

$$\operatorname{Hom}_U(U, E|_U) \overset{def}{=} \operatorname{Hom}_U(U, \operatorname{Spec} \operatorname{Sym}^\bullet(\mathcal{E}^\vee|_U))$$

$$= \operatorname{Hom}_{\mathcal{O}_U\text{-alg}}(\operatorname{Sym}^\bullet(\mathcal{E}^\vee|_U), \mathcal{O}_U)$$

$$= \operatorname{Hom}_{\mathcal{O}_U\text{-mod}}(\mathcal{E}^\vee|_U, \mathcal{O}_U)$$

$\operatorname{Hom}_{A\text{-alg}}(\operatorname{Sym}^\bullet M, B) = \operatorname{Hom}_{A\text{-mod}}(M, B)$ for any A-alg B, A-mod M.

$$= \operatorname{Hom}_{\mathcal{O}_U\text{-mod}}(\mathcal{O}_U, \mathcal{E}|_U)$$

$$= \mathcal{E}(U)$$

Rmk. We often identify locally free sheaves with their associated vector bundles, and use these terms interchangeably. Same for invertible sheaves and line bundles. (From lecture 25 onwards.)

Ex. $X = \mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} A$, $A = \mathbb{C}[x_0, \dots, x_n]$

$$(*) \quad 0 \longrightarrow A(-1) \longrightarrow \bigoplus_{i=0}^n A \longrightarrow Q \longrightarrow 0$$

where Q is defined to be the cokernel

$$1 \longmapsto (x_0, \dots, x_n)$$

$\downarrow \sim$

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X^{\oplus n+1} \longrightarrow \widetilde{Q} \longrightarrow 0$$

$$\mathbb{V}(\mathcal{O}_X^{n+1}) = \operatorname{Spec} \mathcal{O}_X[y_0, \dots, y_n] = \mathbb{P}^n_{\mathbb{C}} \times \mathbb{A}^{n+1}$$

What is $\mathbb{V}(\mathcal{O}_X(-1)) \longrightarrow \mathbb{V}(\mathcal{O}_X^{n+1})$? Dualise $(*)$:

$$0 \longleftarrow A(1) \longleftarrow \bigoplus_{j=0}^n A$$

$$x_i \longleftarrow\!\shortmid\; y_i = (0, \dots, 1, \dots, 0)$$

$$\operatorname{Sym}^\bullet\Big(\bigoplus_{j=0}^n A\Big) = A[y_0, \dots, y_n]$$

the y_i all have $\deg = 0$

$$\operatorname{Sym}^\bullet(A(1)) = A \oplus \underbrace{A(1)}_{\wr} \oplus A(2) \oplus \dots$$

$$= A \oplus \overbrace{At} \oplus At^2 \oplus \dots$$

$$= A[t]$$

where t is a formal symbol of $\deg = -1$ as a graded ring

$$\operatorname{Sym}^\bullet(A(1)) = A[t] \xleftarrow{\operatorname{Sym}^\bullet(\varphi^\vee)} A[y_0, \dots, y_n]$$

$$tx_i \longleftarrow\!\shortmid\; y_i$$

Apply $\sim$ and consider $D(x_0)$.

$$\operatorname{Sym}^\bullet(\mathcal{O}_X^{\oplus n+1})(D(x_0)) = \widetilde{\operatorname{Sym}^\bullet A^{n+1}}(D(x_0)) = \mathbb{C}\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right][y_0, \dots, y_n]$$

$$\operatorname{Sym}^\bullet(\mathcal{O}_X(1))(D(x_0)) = \widetilde{A[t]}(D(x_0)) = (A[t]_{x_0})_0 = \mathbb{C}[x_0, \dots, x_n, t, \tfrac{1}{x_0}]_0 = \mathbb{C}\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}, \underbrace{tx_0}_{=:\lambda}\right]$$

$$\mathbb{V}(\varphi)^{\#} : \mathbb{C}\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right][y_0, \dots, y_n] \longrightarrow \mathbb{C}\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}, \lambda\right]$$

$$\frac{x_i}{x_0} \longmapsto \frac{x_i}{x_0}$$

$$y_0 \longmapsto \lambda$$

$$y_i \longmapsto tx_i = tx_0 \cdot \frac{x_i}{x_0} = \lambda \cdot \frac{x_i}{x_0}$$

Apply Spec:

$$D(x_0) \times \mathbb{A}^1_\lambda \longrightarrow D(x_0) \times \mathbb{A}^{n+1}$$

$$\Big(\underbrace{\left[1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]}_{[x_0, \dots, x_n]}, \lambda\Big) \longmapsto \Big(\left[1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right], \underbrace{\left(\lambda, \frac{x_1}{x_0}\lambda, \dots, \frac{x_n}{x_0}\lambda\right)}\Big)$$

$\{\lambda(x_0, \dots, x_n) \mid \lambda \in \mathbb{C}\}$ line in $\mathbb{P}^n$ defined by $[x_0, \dots, x_n]$

cut out by equations $x_i y_j - x_j y_i$, $\forall i, j$

Conclusion: $\mathbb{V}(\mathcal{O}_X(-1)) \xrightarrow[\text{inversion}]{\text{closed}} \mathbb{V}(\mathcal{O}_X^{n+1}) = \mathbb{P}^n \times \mathbb{A}^{n+1}$

with image $Z = V(x_i y_j - x_j y_i \mid i, j)$

$\underbrace{\mathbb{V}(\mathcal{O}_X(-1))}(\mathbb{C}) = \{(p, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in L_p\}$ L_p is the line def'd by p

called the tautological line bundle. (this is why we needed $\vee$ in the def.)

$$\begin{tikzcd}
\mathbb{P}^n \times \mathbb{A}^{n+1} \\
Z \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\text{projection}", "\pi"'] \arrow[d, "p"', "\text{projection}"] & \mathbb{P}^n \\
\mathbb{A}^{n+1}
\end{tikzcd}$$

$\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$

$$p^{-1}(\lambda) = \begin{cases} [\lambda_0, \dots, \lambda_n] \in \mathbb{P}^n & \text{if } \lambda \neq 0 \\ \mathbb{P}^n & \text{if } \lambda = 0 \end{cases}$$

$\begin{matrix} Z \\ \downarrow \\ \mathbb{A}^{n+1} \end{matrix}$ is called the blowup of $\mathbb{C}^{n+1}$ at the origin.

Algebraic Geometry 1, lecture 25 21.01.2019

Regularity

See [Vakil] or [Liu: AG & Alg. Curves]

Krull's Thm. A noetherian rg, $f_1, \dots, f_r \in A$, Z irred component of $V(f_1, \dots, f_r) \subseteq X = \operatorname{Spec} A$.

Then $\operatorname{codim}(Z, X) \leq r$.

Prop. A noeth local, $k = A/m$. Then $\dim A \leq \dim_k \underbrace{m/m^2}$

Zariski tangent space

Def. A noetherian local ring A is regular if $\dim A = \dim_k m/m^2$

Ex. $\dim A = 0$ then $A \text{ reg} \iff m/m^2 = 0 \overset{\text{NAK}}{\iff} m = 0 \iff A$ is a field.

Ex. $k[x]/_{x^2}$ is not regular

Def. Locally noetherian scheme X is

- regular at $p \in X$ if $\mathcal{O}_{X,p}$ is regular (as a noeth loc ring)
- regular / non-singular if regular at $\forall p \in X$
- singular if not regular.

Warning. regular ≠ smooth. (later)

Ex. 0. Spec k is regular

Ex. 1. $X = \mathbb{A}^1_k = \operatorname{Spec} k[x] = \{(0) = \eta\} \amalg \{(f) \mid f \in k[x] \text{ irreducible}\}$

$$\left.\begin{array}{l} \mathcal{O}_{X,\eta} = k \text{ is regular} \\ \mathcal{O}_{X,(f)} = k[x]_f, \quad m = (f), \quad m/m^2 = (f)/(f^2) \simeq k((f)) \\ \qquad \dim \mathcal{O}_{X,(f)} = 1 = \dim m/m^2 \qquad f \mapsfrom 1 \end{array}\right\} \Rightarrow \mathbb{A}^1_k \text{ is regular}$$

Ex. 2. C curve $/k$, $\mathcal{O}_{C,\eta} = K$ regular

Let $x \in C$ be a closed pt. Is $\mathcal{O}_{C,x}$ regular?

Lemma. A noeth loc rg of dim 1. Then A is regular $\iff$ DVR ($\iff$ integrally closed).

Pf: A DVR $\Rightarrow$ PID $\Rightarrow m/m^2 = (\pi)/(\pi^2) \simeq (A/m)\pi \Rightarrow \dim m/m^2 = 1 = \dim A \to A$ reg.

Conv: A reg $\to m = (\pi)$ by NAK. Wts A PID. Let $I \subseteq A$, $I \neq 0$.

$\bigcap_{l \geq 0} m^l = 0 \Rightarrow \exists n: I \subseteq m^n,\ I \nsubseteq m^{n+1} \Rightarrow \exists y = a \cdot \pi^n \in I$ s.t. $y \notin m^{n+1} \Rightarrow a \notin m$

$\Rightarrow a \in A^\times \Rightarrow a^{-1} y = \pi^n \in I \Rightarrow m^n = (\pi^n) \subseteq I \subseteq m^n \Rightarrow I = (\pi^n)$

Moreover, $\forall y_1, y_2 \in A$: $y_1 = a_1 \pi^{n_1}$; $y_2 = a_2 \pi^{n_2}$ where $a_1, a_2 \in A^\times \Rightarrow y_1 \cdot y_2 = \underbrace{a_1 a_2}_{\in A^\times} \cdot \underbrace{\pi^{n_1 + n_2}}_{\neq 0} \neq 0$.

$\Rightarrow$ A is a domain.

$\Rightarrow$ C is regular iff normal

Hard Fact. (CA) A regular noetherian local ring. Then

a) A is an integral domain (easy)

b) $\forall \mathfrak{p} \in \operatorname{Spec} A$: $A_\mathfrak{p}$ is regular.

c) A is factorial, hence normal.

Cor. A noetherian scheme X is regular $\Leftrightarrow$ regular at its closed pts.

Pf: Let $p \in X$. X qc $\Rightarrow \exists\, q \in \overline{\{p\}}$ closed pt.

$\exists\, U := \operatorname{Spec} A$ s.t. $q \in U \Rightarrow p \in U$.

$q \in \overline{\{p\}} \Leftrightarrow p \subseteq q$ as ideals in A $\Rightarrow A_p = (A_q)_{pA_q}$

b) $\Rightarrow A_q$ regular $\Rightarrow A_p$ regular. □

Thm. (Jacobian criterion of regularity)

k field, $X = \operatorname{Spec}\left(k[x_1, \dots, x_n] / (f_1, \dots, f_r)\right) \subseteq \mathbb{A}^n_k$

$p \in X$ s.t. $k(p) = k$, (p corresponds to $(x_1 - a_1, \dots, x_n - a_n)$)

Let

$$\mathcal{J}_p := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix} \in M_{n \times r}(k)$$

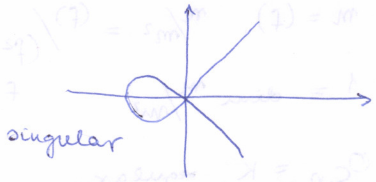
Then f is regular at p $\Leftrightarrow$ $\operatorname{rk} \mathcal{J}_p = n - \dim \mathcal{O}_{X,p}$.

Cor. $k = \overline{k}$. Then $X = V(f)$ reg $\Leftrightarrow$ the system $\left\{ f = 0, \frac{\partial f}{\partial x_i} = 0 \;\middle|\; i \right\}$ has no solution in k^n. □

$\uparrow$

HF + Thm.

Ex. $X = V(y^2 = x^2(x+1))$, $f = y^2 - x^3 - x^2$



Pf: Wma $p = (0, \dots, 0)$. by linear change of coordinates.

$\mathfrak{m} := (x_1, \dots, x_n)$ ideal of p in $\mathbb{A}^n_k$

$\mathfrak{n} := \mathfrak{m}/I$ ideal of p in X where $I = (f_1, \dots, f_r)$

$\delta x_i :=$ img of x_i in $\mathfrak{m}/\mathfrak{m}^2$ $\Rightarrow$ $\delta x_1, \dots, \delta x_n$ basis of $\mathfrak{m}/\mathfrak{m}^2$

Consider

$$\begin{tikzcd} I \arrow[r, hook] \arrow[rr, bend left, "D"] & \mathfrak{m} \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \simeq k\delta x_1 \oplus \dots \oplus k\delta x_n \end{tikzcd}$$

$D(g) = (\text{linear terms in } g) = \frac{\partial g}{\partial x_1}(0)\, \delta x_1 + \dots + \frac{\partial g}{\partial x_n}(0)\, \delta x_n$

(E.g. $D(x_1 + 3x_2 + x_1^2) = \delta x_1 + 3\delta x_2$)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & I \arrow[r] \arrow[d] \arrow[dr, "D"] & I/_{I \cap m^2} \arrow[d] & \\
0 \arrow[r] & m^2 \arrow[r] \arrow[d] & m \arrow[r] \arrow[d] & m/_{m^2} \arrow[r] \arrow[d] & 0 \\
 & n^2 \arrow[r] \arrow[d] & n \arrow[r] \arrow[d] & n/_{n^2} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Get: $I \xrightarrow{D} m/_{m^2} \longrightarrow n/_{n^2} \longrightarrow 0$ exact, $n/_{n^2} \simeq \operatorname{Coker} D \simeq \operatorname{Coker} \mathcal{J}_p$

$\Rightarrow \dim n/_{n^2} = n - \operatorname{rk} \mathcal{J}_p$

$= \dim \mathcal{O}_{X,p}$ iff $\operatorname{rk} \mathcal{J}_p = n - \dim \mathcal{O}_{X,p}$

Problems

Ex. 1. $k \neq \bar{k}$ then the criterion may not be useful:

$k = \mathbb{F}_p(u)$, $X = V(f := x^p - u) \subseteq \mathbb{A}^1_k$ $\quad \frac{\partial f}{\partial x} = p x^{p-1} - 0 = 0$, but $k[x]/_{x^p - u} =: L$ is a field as $(x^p - u)$ is irreducible, thus $X = \operatorname{Spec} L$ is regular.

Ex. 2. Regularity is not preserved by base change or field extensions.

Let X be as in Ex. 1., $L := k[y]/_{(y^p - u)}$

$X \times_k L = L[x]/_{(x^p - u)} = L[x]/_{(x^p - y^p)} = L[x]/_{(x-y)^p}$ not reduced

Ex. 3. $\exists X, Y$ varieties $/k$ s.t. X, Y are both regular but $X \times_k Y$ is not regular. (See Ex. 2 for a possible choice of X, Y)

Kähler differentials

A ring, B an A-algebra, M a B-module

Def. A-derivation of B into M: an A-linear $\varphi : B \to M$ s.t. $\varphi(ab) = a\varphi(b) + b\varphi(a)$ $\forall a, b \in B$

Ex. $\varphi(1) = \varphi(1 \cdot 1) = 1 \cdot \varphi(1) + 1 \cdot \varphi(1) = 2 \cdot \varphi(1) \Rightarrow \varphi(1) = 0 \Rightarrow \varphi(a) = 0$ $\forall a \in A$.

Ex. Y smooth manifold, $p \in Y$, $B = C^\infty(Y)$, $M = A = \mathbb{R}$, $\quad B \times M \longrightarrow M$, $(f, a) \longmapsto f(p) \cdot a$

$\{A\text{-derivations}\} \simeq T_p Y$ tangent space

$$\frac{\partial}{\partial x}(fg)_p = f(p) \frac{\partial g}{\partial x}(p) + g(p) \frac{\partial f}{\partial x}(p)$$

Notation. $\operatorname{Der}_A(B, M) := \{$ A-derivations of B into M $\}$

Consider the functor

$$\mathrm{Der}_A(B,-) : B\text{-mod} \longrightarrow \mathrm{Set}$$
$$M \longmapsto \mathrm{Der}_A(B,M)$$
$$(g: M \to M') \longmapsto \left(\begin{array}{c} \mathrm{Der}_A(B,M) \longrightarrow \mathrm{Der}_A(B,M') \\ g \longmapsto \varphi \circ g \end{array}\right)$$

Prop. $\mathrm{Der}_A(B,-)$ is corepresentable, i.e. $\exists!\ \Omega_{B/A}$ B-module and $\delta : B \to \Omega_{B/A}$ derivation:

$$\mathrm{Hom}_{B\text{-mod}}(\Omega_{B/A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B,M) \quad \text{iso} \quad \forall M \ B\text{-module}.$$
$$g \longmapsto g \circ \delta$$

Ex. (ctd.) $\mathrm{Hom}_{B\text{-mod}}(\Omega_{B/k}, k) \simeq \mathrm{Der}_k(B,k) = T_pY$

$\wr\wr$

$\mathrm{Hom}_{B\text{-mod}}(\underbrace{\Omega_{B/k} \otimes k}_{\text{cotangent space to } p \in Y}, k)$

24.01.2019

Prop. The same as above, restating: $\forall \varphi : B \to M$ A-derivation $\exists!\ g : \Omega_{B/A} \to M$ s.t. $g \circ \delta = \varphi$.

Pf: $F := \bigoplus_{x \in B} B.\delta_x$ free B-module, δ_x a symbol $\forall x \in B$

$I := \langle \delta(ab) - \delta(a)b - a\delta(b),\ \delta(a+b) - \delta(a) - \delta(b),\ \delta(c) \mid a,b \in B,\ c \in A \rangle \leq F$ submodule

$\Omega_{B/A} := F/I$, $\quad \delta : B \longrightarrow \Omega_{B/A}$
$\qquad b \longmapsto \delta_b$

Universal property: Let $\varphi : B \longrightarrow M$ be an A-derivation.

$\tilde{g} : F \longrightarrow M$
$\quad \delta_x \longmapsto \varphi(x)$

Then $\tilde{g}(I) = 0$ since φ is a derivation, hence $\tilde{g}$ descends to $g : F/I = \Omega_{B/A} \longrightarrow M$,

and by construction we have $g(\delta(x)) = g(\delta_x) = \varphi(x) \Rightarrow g \circ \delta = \varphi$.

Uniqueness: comes from the fact that δ_x $(x \in B)$ generate $\Omega_{B/A}$. □

Ex. $B := A[x_1, \dots, x_n]$

Claim. $\Omega_{B/A} = B\delta x_1 \oplus \dots \oplus B\delta x_n$, $\quad \delta : B \longrightarrow \Omega_{B/A}$
$$f \longmapsto \sum_i \frac{\partial f}{\partial x_i} \delta x_i$$

Checking UP: let $\varphi : B \to M$ A-derivation. $\Rightarrow \varphi(f) = \sum_i \frac{\partial f}{\partial x_i} \varphi(x_i)$

Set $g : \Omega_{B/A} \longrightarrow M$ $\quad \Rightarrow g \circ \delta = \varphi$. ✓
$\qquad \delta x_i \longmapsto \varphi(x_i)$

Ex. $B = A/I \Rightarrow \varphi = 0 \ \forall \varphi \in \mathrm{Der}_A(B,M) \Rightarrow \Omega_{B/A} = 0$

Ex. $B = S^{-1}A$, $\varphi \in \mathrm{Der}_A(B, M)$. $\forall b \in B \ \exists s \in S \subseteq A$: $bs \in A$.

$\Rightarrow \quad 0 = \varphi(bs) = \varphi(b)s + b\underbrace{\varphi(s)}_{0} = \varphi(b)\cdot s$. Since s is invertible: $\varphi(b) - 0$

$\Rightarrow \quad \varphi = 0 \quad \Rightarrow \quad \Omega_{B/A} = 0$

Ex. A' and B A-algebras. $\Rightarrow$

$$\begin{tikzcd} B' := B \otimes_A A' & B \arrow[l] \\ A' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$\to \Omega_{B/A} \otimes_B B' \simeq \Omega_{B'/A'}$ canonical isomorphism

Pf: Let $\delta: B \longrightarrow \Omega_{B/A}$, apply $\otimes B'$, get $\delta' : B' \longrightarrow \Omega_{B/A} \otimes B'$.

Checking UP: let $\varphi \in \mathrm{Der}_{A'}(B', M)$. $\Rightarrow$ $B \to B' \longrightarrow M_B$ is an A-derivation where M_B is M considered as a B-module.

Let $g: \Omega_{B/A} \longrightarrow M_B$ be the corresponding homomorphism,
$\delta b \longmapsto \varphi(b)$

and $\tilde{g} : \Omega_{B/A} \otimes B' \longrightarrow M$ the induced homomorphism.
$\delta b \otimes b' \longmapsto b'\varphi(b)$

Then $\tilde{g} \circ \delta' = \varphi$.

Prop. Suppose $A \longrightarrow B \longrightarrow C$ is exact.

a) $\Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \longrightarrow 0$ is exact.

b) If $C = B/I$ then $I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes C \xrightarrow{\alpha} \Omega_{C/A} \longrightarrow 0$ is exact.

Here the morphisms are given as follows:

- $B \longrightarrow C \xrightarrow{\delta} \Omega_{C/A}$ is an A-derivation of B into $(\Omega_{C/A})_B$.
  Let $g: \Omega_{B/A} \to (\Omega_{C/A})_B$, and $\alpha: \Omega_{B/A} \otimes C \longrightarrow \Omega_{C/A}$ be the induced map.
  $\delta_B b \otimes c \longmapsto c\,\delta_C b$
- $\delta : C \longrightarrow \Omega_{C/B}$ is an A-derivation (every B-derivation is an A-derivation), and we get an induced morphism $\beta: \Omega_{C/A} \longrightarrow \Omega_{C/B}$.

Pf: a) For any C-module N, apply $\mathrm{Hom}_C(-, N)$:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_C(\Omega_{C/B}, N) \arrow[r, "\beta^\vee"] \arrow[d, phantom, "\wr"] & \mathrm{Hom}_C(\Omega_{C/A}, N) \arrow[r, "\alpha^\vee"] \arrow[d, phantom, "\wr"] & \mathrm{Hom}_C(\Omega_{B/A} \otimes C, N) \arrow[d, phantom, "\wr"] \\ & \mathrm{Der}_B(C, N) \arrow[r] & \mathrm{Der}_A(C, N) \arrow[dr] & \mathrm{Hom}_B(\Omega_{B/A}, N_B) \arrow[d, phantom, "\wr"] \\ & \varphi \arrow[r, mapsto] & \varphi \arrow[dr, mapsto] & \mathrm{Der}_A(B, N_B) \\ & & & \varphi|_B \end{tikzcd}$$

$\alpha^\vee(\varphi) = 0 \in \mathrm{Der}_A(C, N) \iff \varphi(b) = 0 \quad \forall b$

$\iff \varphi$ is a B-derivation

$\iff \varphi \in \mathrm{Im}(\beta^\vee)$

b) $C = B/I$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_C(\Omega_{C/A}, N) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}_C(\Omega_{B/A} \otimes C, N) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}_C(\overbrace{I/I^2}^{\simeq I \otimes_B C}, N) \arrow[d, phantom, "\cong" rotate=90] \\
& \mathrm{Der}_A(C, N) \arrow[r] & \mathrm{Der}_A(B, N_B) \arrow[r] & \mathrm{Hom}_B(I, N) \\
& \varphi \arrow[r, mapsto] & \varphi|_B \quad \varphi \arrow[r, mapsto] & \varphi|_I
\end{tikzcd}$$

What is $I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes C$?

$$\begin{tikzcd}
f, g \in I \arrow[r] \arrow[d] \arrow[rrr, bend right, "\bar{\delta}"'] & B \arrow[r, "\delta"] & \Omega_{B/A} \arrow[r] & \Omega_{B/A} \otimes_B C \\
a \in I/I^2 \arrow[rrru, bend right, "\exists ! \hat{\delta}"'] & & &
\end{tikzcd}$$

$\bar{\delta}(fg) = \underbrace{f}_{\in I} \bar{\delta}(g) + \bar{\delta}(f) \underbrace{g}_{\in I} = 0$ in $C = B/I$,

$\hat{\delta}(ba) = \hat{\delta}(b) \underbrace{a}_{\in I} + b\hat{\delta}(a) = b\hat{\delta}(a)$ in C

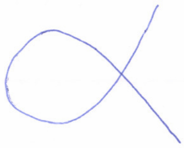
$\Rightarrow \hat{\delta}$ is a B-module morph.

Let $\varphi \in \mathrm{Der}_A(B, N_B)$ s.t. $\varphi|_I = 0$. Then $\exists \bar{\varphi} : C \to N$ s.t. $\bar{\varphi}|_B = \varphi$

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[d] & N \\
B/I \arrow[ru, "\bar{\varphi}"'] &
\end{tikzcd}$$

□

Ex. for b): $A := k$, $B := k[x, y]$, $C := k[x,y]/_{y^2 = x^2(x+1)}$, $f = y^2 - x^3 - x^2$



Spec C

$$\left((f)/(f^2) = I/I^2 \right) \xrightarrow{\delta} \left(\Omega_{B/A} \otimes C = C\delta x \oplus C\delta y \right) \longrightarrow \Omega_{C/k} \longrightarrow 0$$

$$f \longmapsto \delta f = (-3x^2 - 2x)\delta x + 2y\delta y$$

$\Omega_{B/A} = B\delta x \oplus B\delta y$ $\Rightarrow \Omega_{C/k} = \left(C\delta x \oplus C\delta y\right) \Big/ \left((3x^2 + 2x)\delta x - 2y\delta y\right)$

If $y \neq 0$ or $3x^2 + 2x \neq 0$ (while $f = 0$) then $\Omega_{C/k}$ is loc free of rk 1.

Other: $y = 0$, $3x^2 + 2x = 0$, $f = 0$ $\Leftrightarrow$ $(x, y) = 0$ assuming char $k \neq 2, 3$

$\Rightarrow \Omega_{C/k}$ is loc free outside of the origin.

Ex. for a) $A := k$, $C := k[x, y]/_{y^2 = x^2(x+1)}$ (as before), $B := k[x] \hookrightarrow C$

$$\left(\Omega_{B/k} \otimes C = C\delta x\right) \longrightarrow \Omega_{C/k} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

$$\Omega_{C/k} = \left(C\delta x \oplus C\delta y\right) \Big/ \left((3x^2 + 2x)\delta x = 2y\delta y\right)$$

$$\Rightarrow \Omega_{C/B} = C\delta y / 2y\delta y = \left(C/(2y) \right) \delta y = k[x]/(x^3 - x^2)\, \delta y$$

Supported at $y = 0$, $x = 0$ (length 2), $x = -1$ (length -1).

Global construction

Key Prop. Let B an A-algebra, $\mu : B \underset{A}{\otimes} B \longrightarrow B$

$b_1 \otimes b_2 \longmapsto b_1 b_2$

Let $I := \operatorname{Ker} \mu$ (viewed as a left B-module), $\delta : B \longrightarrow I/I^2$ $\Rightarrow$ $\left(I/I^2, \delta\right) \simeq \left(\Omega_{B/A}, \delta\right)$

$b \longmapsto 1 \otimes b - b \otimes 1$

PF: δ is a derivation: $0 = (1 \otimes b - b \otimes 1)(1 \otimes a - a \otimes 1) \in I/I^2$, expand. ✓

Universal property: let $\varphi \in \operatorname{Der}_A(B, M)$, $\alpha : B \underset{A}{\otimes} B \longrightarrow M$

$b_1 \otimes b_2 \longmapsto b_1 \varphi(b_2)$

$$\begin{aligned}
\alpha\big((b_1 \otimes b_2)(c_1 \otimes c_2)\big) &= \alpha(b_1 c_1 \otimes b_2 c_2) \\
&= b_1 c_1 \, \varphi(b_2 c_2) \\
&= b_1 c_1 \big(b_2 \varphi(c_2) + \varphi(b_2) c_2\big) \\
&= b_1 b_2 \, c_1 \varphi(c_2) + c_1 c_2 \, b_1 \varphi(b_2) \\
&= \mu(b_1 \otimes b_2)\, \alpha(c_1 \otimes c_2) + \mu(c_1 \otimes c_2)\, \alpha(b_1 \otimes b_2)
\end{aligned}$$

$\rightarrow$ $\forall \gamma, \gamma' \in B \underset{A}{\otimes} B$: $\alpha(\gamma \cdot \gamma') = \mu(\gamma)\, \alpha(\gamma') + \alpha(\gamma)\, \mu(\gamma')$

$\rightarrow$ $\alpha|_{I^2} = 0$, get an induced morphism $\bar{\alpha} : I/I^2 \longrightarrow M$

$\bar{\alpha}(\delta(b)) = \bar{\alpha}(1 \otimes b - b \otimes 1) = 1 \cdot \varphi(b) - b \underbrace{\varphi(1)}_{0} = \varphi(b)$, i.e. $\bar{\alpha} \circ \delta = \varphi$. ✓

Def. Let $f : X \longrightarrow Y$ be a morphism of schemes, $\Delta : X \longrightarrow X \underset{Y}{\times} X$ a locally closed subscheme, $\mathcal{J} := \operatorname{Ker}\left(\mathcal{O}_{X \times_Y X} \xrightarrow{\mu} \Delta_* \mathcal{O}_X\right)$. Then $\Omega_{X/Y} := \Delta^*\left(\mathcal{J}/\mathcal{J}^2\right)$ is the sheaf of Kähler differentials.

Rmk. $f : X \longrightarrow Y$, $\mathcal{F}$ an $\mathcal{O}_Y$-module, $f^* \mathcal{F} := f^{-1} \mathcal{F} \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$. pullback

Recall that $f^{-1}\mathcal{F} = \left(U \mapsto \varinjlim_{f(U) \subseteq V} \mathcal{F}(V)\right)^{sh}$.

If $\mathcal{F}$ is qcoh, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $\mathcal{F} = \tilde{M}$, M a B-module then $f^* \mathcal{F} = \left(M \underset{B}{\otimes} A\right)^{\sim}$

Rmk. $f : X \longrightarrow Y$, $U = \operatorname{Spec} A \subseteq X$, $V = \operatorname{Spec} B \subseteq f^{-1}(U)$ open affines.

$U \underset{V}{\times} U = \operatorname{Spec}\left(B \underset{A}{\otimes} B\right) \subseteq X \underset{Y}{\times} X$ open affine

$\Delta(X) \cap \left(U \underset{V}{\times} U\right) = V(I)$ where $I = \operatorname{Ker}\left(B \underset{A}{\otimes} B \longrightarrow B\right)$ $\rightarrow$ $\mathcal{J}\big|_{U \times_V U} = \tilde{I}$

$\Omega_{X/Y}\big|_V = \Delta\big|_V^*\left(\mathcal{J}/\mathcal{J}^2\big|_V\right) = \Delta^*\left(\widetilde{I/I^2}\right) \underset{\uparrow}{=} \left(I/I^2 \underset{B \otimes_A B}{\otimes} B\right)^{\sim} = \widetilde{I/I^2}$ where I/I^2 is viewed as a B-module

prev Rmk.

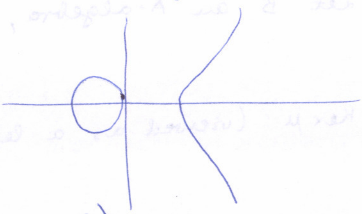
$\overset{KP}{=} \widetilde{\Omega_{B/A}}$

Upshot. $\Omega_{X/Y}$ is the global version of $\Omega_{B/A}$ and it is quasicoherent.

28.01.2019

<u>Ex.</u> Let char $k \neq 2$, $C \subseteq \mathbb{P}^2_k$ given by $ZY^2 = (X+Z)Y(X-Z)$

On $D(Z)$, $x := \frac{X}{Z}$, $y := \frac{Y}{Z}$ $\Rightarrow C \cap D(Z) = V(y^2 = x^3 - x)$, we have already seen this



<u>Prop.</u> $\exists \omega \in \Gamma(C, \Omega_{C/k})$ s.t.
$$\begin{tikzcd} \mathcal{O}_C \arrow[r] & \Omega_{C/k} \\ 1 \arrow[r, mapsto] & \omega \end{tikzcd}$$
is an iso.

In part., $\Gamma(C, \Omega_{C/k})$ if $k = \bar{k}$ (this assumption may be removed)

Pf: $A := k[x,y]\big/(y^2 = x^3 - x)$

$$\Omega_{C \cap D(Z)/k} = (A\delta x \oplus A\delta y)/\delta f = (A\delta x \oplus A\delta y)\big/(2y\,\delta y = (3x^2-1)\,\delta x)$$

On $U_1 = D(y)$, set $\omega_1 := \frac{\delta x}{y} \in \Gamma(U_1, \Omega_{C/k})$

On $D(y) \cap D(3x^2-1)$: $\frac{\delta x}{y} = \frac{1}{y} \cdot \frac{2y}{3x^2-1}\delta y = \overset{\text{recall char } k \neq 2}{\frac{2}{3x^2-1}}\delta y$

On $U_2 = D(x)$, set $\omega_2 := \frac{2}{3x^2-1}\delta y$ $\Rightarrow \omega_1\big|_{D(y)\cap D(3x^2-1)} = \omega_2\big|_{D(y)\cap D(3x^2-1)}$

On $D(Y)$, let $t := \frac{X}{Y}$, $u := \frac{Z}{Y}$ $\longrightarrow x = \frac{t}{u}$, $y = \frac{1}{u}$

$D(Y) \cap C = V(u = t^3 - tu^2)$, $B := k[u]\big/(u = t^3 - tu^2)$

$$\Omega_{D(Y)\cap C/k} = (B\delta t \oplus B\delta u)\big/\big((1+2ut)\delta u = (3t^2 - u^2)\delta t\big)$$

On $D(Y) \cap D(Z) \cap D(1+2ut)$: $\frac{\delta x}{y} = u\,\delta\left(\frac{t}{u}\right) = \delta t - \underset{\text{we don't like this pole at } u}{\frac{t}{u}}\delta u = \delta t - \frac{t}{u} \cdot \frac{3t^2 - u^2}{1+2ut} \cdot \delta u =$

$$\overset{t^3 = tu^2 + u}{=} \delta t \left(1 - \frac{3tu^2 + u - tu^2}{u(1+2ut)}\right) = \frac{4}{1+2ut} \cdot \delta t$$ extends to $t = u = 0$

On $U_3 = D(1+2ut)$ set $\omega_3 := \frac{4}{1+2ut}\delta t \Rightarrow \omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$ $\forall i,j \in \{1,2,3\}$,

and $C = U_1 \cup U_2 \cup U_3$. $\Rightarrow$ get $\omega \in \Gamma(C, \Omega_{C/k})$ s.t. $\omega|_{U_i} = \omega_i$

It follows from these calculations that $\Omega_{C/k}$ is locally free of rank 1,

e.g. on U_3, it is $(B\delta t \oplus B\delta u)/\ldots \cong B\delta t$

Moreover $\omega(x) \neq 0$ in $\Omega_{C/k,x} \underset{\mathcal{O}_{C,x}}{\otimes} k(x)$ $\forall x \in C$

$$\Rightarrow \begin{tikzcd} \mathcal{O}_C \arrow[r] & \Omega_C \\ 1 \arrow[r, mapsto] & \omega \end{tikzcd}$$ is an iso on stalks, hence iso.

The last assertion follows from the following

<u>Fact.</u> For a proper variety X over $k = \bar{k}$, $\Gamma(X, \mathcal{O}_X) = k$.

Pf:
$$\begin{tikzcd} \mathrm{Hom}_{Sch/k}(X, \mathbb{A}^1_k) \arrow[r, "\sim"] & \mathrm{Hom}_{k\text{-alg}}(k[t], \mathcal{O}_X(X)) \cong \mathcal{O}_X(X) \\ (f : X \to \mathbb{A}^1_k) \arrow[r, mapsto] & f^\#(t) \end{tikzcd}$$

Algebraic Geometry I, Lecture 27

Let $f: X \longrightarrow \mathbb{A}^1_k$. Since X is proper and $\mathbb{P}^1_k$ is separated, f is proper by the upcoming Lemma.

move

$\Rightarrow \iota \circ f : X \longrightarrow \mathbb{A}^1_k \xhookrightarrow{\iota} \mathbb{P}^1_k$ is proper since ι is an open immersion

$(\iota \circ f)(X)$ is closed in $\mathbb{P}^1_k$ and $\neq \mathbb{A}^1_k \Rightarrow f(X) = \{(x-a)\}$ for some $a \in k$

$\Rightarrow f^*(x-a)(p) = 0 \quad \forall p \in X \Rightarrow f^*(t-a)^N = 0 \quad \forall N \geq 1 \Rightarrow f^*(t-a) = f^*(t) - a = 0.$

Lemma. $X \xrightarrow{f} Y \xrightarrow{g} Z$, $g \circ f$ proper & g separated $\Rightarrow$ f proper.

Pf: Factor f as $pr_Z \circ \underbrace{(id, f)}_{\text{graph of } f}$.

Rmk. $k = \bar{k}$ is necessary: e.g. $X = \operatorname{Spec} \mathbb{C} = \operatorname{Spec} \mathbb{R}[x]/(x^2+1)$, X is a proper vty., $\mathcal{O}_X(X) = \mathbb{C}$ and not $\mathbb{R}$

Def. C normal projective curve $/k$. The genus of C is $g(C) := \dim_k \Gamma(C, \Omega_{C/k})$

Ex. $g(C) = 1$ in the case of the Prop.

Ex. $\mathbb{P}^1_k =: C$, $\Omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$ exc. $\Rightarrow \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = \Gamma(\mathbb{P}^1, \mathcal{O}(-2)) = 0 \Rightarrow g(C) = 0.$

Def. A scheme X/k is smooth of dimension n over k if

- X is locally of finite type
- X is pure of dimension n, i.e. every irred component is of dimension n
- $\Omega_{X/k}$ is locally free of rank n.

Ex. $X := \operatorname{Spec} k[x]/x^2$ is not smooth:

$$\Omega_{X/k} = \left(k[x]/_{x^2}\right) \delta x \Big/ \underbrace{2x\,\delta x}_{=\delta(x^2)} = \begin{cases} k\,\delta x & \text{if char } k \neq 2 \quad \rightarrow \text{ not free} \\ k[x]/_{x^2}\,\delta x & \text{if char } k = 2 \quad \rightarrow \text{ free of wrong rank} \end{cases}$$

Ex. Lemma. K/k fin extension. Then $X = \operatorname{Spec}(K)$ is smooth $/k \Leftrightarrow K/k$ sep'ble.

By this Lemma, $X = \operatorname{Spec}(K)$ is always smooth over K but may fail to be smooth $/k$.

Pf: $\forall \alpha \in K$: let f_α be the minimal polynomial.

$0 = \delta 0 = \delta f_\alpha(\alpha) = f'_\alpha(\alpha) \cdot \delta\alpha$

If K/k is sep'ble $\rightarrow \delta\alpha = 0 \quad \forall \alpha \rightarrow \Omega_{K/k} = 0$, the converse is similar.

Def. X is smooth over k if every component is smooth of some dimension $/k$.

Rmk/Def. $\mathcal{T}_X := \Omega^\vee_{X/k} \overset{def}{=} \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ is the tangent sheaf

If $\Omega_{X/k}$ isn't locally free of fin rk then we may have $\Omega^{\vee\vee}_{X/k} \not\cong \Omega_{X/k}$, $\Omega_{X/k}$ somehow "contains more information" than $\Omega^\vee_{X/k}$.

Def. The vector bundle associated to T_X resp $\Omega_{X/k}$ is the tangent resp. cotangent bundle of X for X smooth.

Def. $\omega_X := \Lambda^n \Omega_{X/k}$ is the canonical line bundle where X is smooth of dim n /k.

This is like the sheaf of holomorphic volume forms.

Ex. $X = \operatorname{Spec} A = \mathbb{A}^n$, $A = k[x_1, \dots, x_n]$, $\Omega_{X/k} = A\delta x_1 \oplus \dots \oplus A\delta x_n$,

$\omega_X = A\, \delta x_1 \wedge \dots \wedge \delta x_n$

Regular vs. smooth

Thm. X scheme oft /k.

a) X smooth /k $\Rightarrow$ X regular.

b) If k is perfect then X smooth /k $\Leftrightarrow$ X regular.

Recall. A not necessarily algebraic K/k is sep'ble if $\exists \{x_i\}$ transcendence base s.t. $K/k(\{x_i\})$ is sep'ble. alg extn. A field k is perfect iff all $\overset{\text{fin.gen.}}{\text{extensions}}$ are sep'ble. Note that fields of char 0 and alg closed fields are perfect.

Key Lemma. X scheme oft /k, $p \in X$ s.t. $k(p) = k$. Then $\Omega_{X,p} \underset{\mathcal{O}_{X,p}}{\otimes} k(p) = \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subseteq \mathcal{O}_{X,p}$ max.

Algebraic version: B noeth. loc. ring, $B/\mathfrak{m} = k$, $k \subseteq B$. Then $\Omega_{B/k} \underset{B}{\otimes} k \cong \mathfrak{m}/\mathfrak{m}^2$

Pf: $A := k$, $C := B/\mathfrak{m} = k$, $A \longrightarrow B \longrightarrow C$

$$\Rightarrow \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \underset{B}{\otimes} k \longrightarrow \underbrace{\Omega_{C/k}}_{= \Omega_{k/k} = 0} \longrightarrow 0 \quad \Rightarrow \delta \text{ is surjective.}$$

So to prove the lemma, wts δ is injective. Equivalently: wts $\underset{\text{dual map}}{\delta^*}$ is surjective.

$$\begin{tikzcd} \mathfrak{m} \arrow[r] \arrow[d] & B \arrow[r, "\delta"] & \Omega_{B/k} \arrow[r] & \Omega_{B/k} \underset{B}{\otimes} k \\ \mathfrak{m}/\mathfrak{m}^2 \arrow[rrru, "\delta"'] & & & \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Hom}_k(\Omega_{B/k} \underset{B}{\otimes} k, k) \arrow[r, "\delta^*"] \arrow[d, phantom, "\cong" sloped] & \operatorname{Hom}(\mathfrak{m}/\mathfrak{m}^2, k) \\ \operatorname{Hom}_B(\Omega_{B/k}, k) \arrow[d, phantom, "\cong" sloped] & \\ \operatorname{Der}_k(B, k) & \end{tikzcd}$$

Let $h : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow k$,

and define $\varphi : B \longrightarrow k$
$b \longmapsto h(b - b_0)$

where $(-)_0$ is the img in k.

Given $a, b \in B$: $0 = h(\underbrace{(a - a_0)(b - b_0)}_{\in \mathfrak{m}^2}) = h(ab - (ab)_0) -$
$- a h(b - b_0) - b_0 h(a - a_0) = \varphi(ab) - a_0 \varphi(b) - b_0 \varphi(a)$

$\Rightarrow$ φ derivation /k corresponding to $g : \Omega_{B/k} \to k$.

$$\delta^*(g)(\underset{\mathfrak{m}}{\overset{\cap}{c}}) = g(\delta c) = \varphi(c) = h(c - c_0) = h(c)$$

PF OF THM (6) WHEN $k = \bar{k}$:

Assume X is connected and smooth of dim n /k.

X pure of dim n $\overset{X \text{ aff}}{\Longrightarrow}$ $\dim \mathcal{O}_{X,p} = n$ $\forall p \in X$ closed pt

$\Omega_{X/k}$ is loc free of rk n. Let $p \in X$ be closed. Then $k(p) = k$, since $k = \bar{k}$.

$$\Rightarrow \mathfrak{m}_p / \mathfrak{m}_p^2 = \Omega_{X/k,\,p} \underset{\mathcal{O}_{X,p}}{\otimes} k(p) \quad \text{by Lemma}$$

$$\cong \mathcal{O}_{X,x}^{\oplus n} \otimes k(p) = k(p)^{\oplus n}$$

$\forall p \in X$ closed: $\dim \mathfrak{m}/\mathfrak{m}^2 = n = \dim \mathcal{O}_{X,p} \Rightarrow p \in X$ regular.

Hard Fact from before: if X is regular at every closed point then it is regular.

Conversely, let X be regular. Then $\forall p \in X$: $\mathcal{O}_{X,p}$ is regular $\Rightarrow$ integral

$\Rightarrow X$ is integral of some dimension D.

Assume $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r) =: \operatorname{Spec} A$, let $I := (f_1, \dots, f_r) \subseteq A$.

$$\Rightarrow \tilde{I}/\tilde{I}^2 \xrightarrow{\delta} \Omega_{\mathbb{A}^n/X} \longrightarrow \Omega_{X/k} \longrightarrow 0$$

$$\begin{tikzcd}
f_i & I/I^2 \arrow[r, "\delta"] & A\delta x_1 \oplus \dots \oplus A\delta x_n \arrow[r] & \Omega_{A/k} \arrow[r] & 0 \\
e_i \arrow[u, mapsto] & A^{\oplus r} \arrow[u] \arrow[ur, "\mathcal{J}_{(f_1, \dots, f_r)}\ \text{Jacobian}"'] & & &
\end{tikzcd}$$

Jacobian criterion $\Rightarrow$ $\forall p \in X$ closed: $\operatorname{rk}(\mathcal{J}_p) = n - \dim \mathcal{O}_{X,p} = n - D$

$V((n-D) \times (n-D) \text{ minors of } \mathcal{J})$, $D((n-D+1) \times (n-D+1) \text{ minors of } \mathcal{J})$ contain no closed points $\rightarrow$ empty $\Rightarrow \operatorname{rk} \mathcal{J}_p = n - D$ $\forall p \in X$

$$\dim\Big(\Omega_{X/k} \underset{\mathcal{O}_{X,p}}{\otimes} k(p)\Big) = n - \operatorname{rk} \mathcal{J}_p = D \quad \text{independent of } p$$

$\Rightarrow \Omega_{X/k}$ loc free of rk D

In the perfect case, we reduce to the alg closed case:

- X smooth /k $\Leftrightarrow$ $X \times_k \bar{k}$ smooth /$\bar{k}$
- k perfect, X regular $\Leftrightarrow$ $X \times_k \bar{k}$ regular.

Question. Let X be a singular vty $/k$. Can we find a desingularisation of X, i.e. proper birational $X' \longrightarrow X$ s.t. X' is non-singular?

Answer. char $k = 0 \longrightarrow$ yes (Hironaka)

char $k \neq 0 \longrightarrow$ open

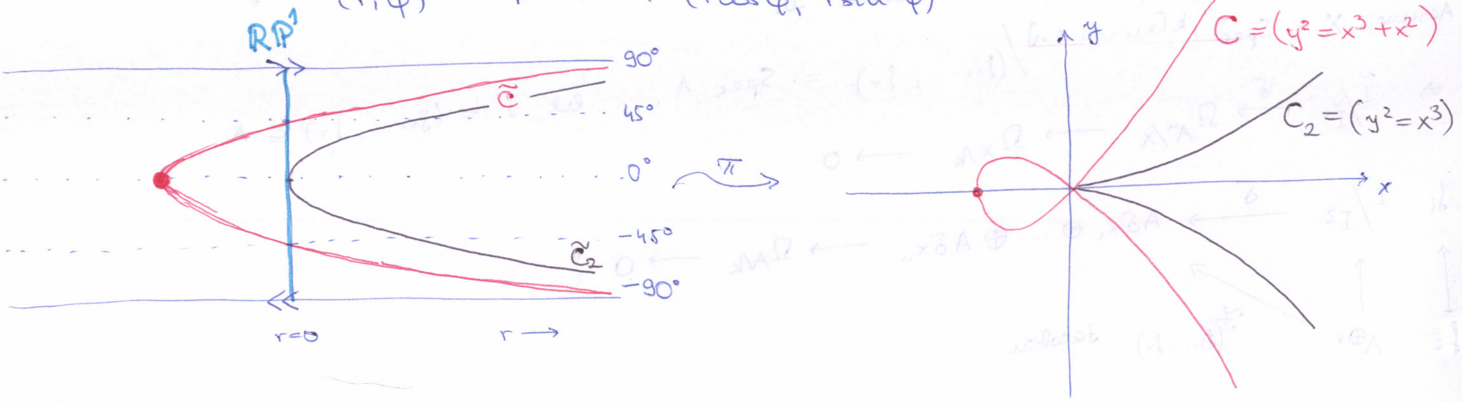
Ex. $V(xy - z^2) \subseteq \mathbb{A}^3$ ~~sing~~ $\longrightarrow$ regularity and normality may not be the same (see PS 10). In dim 1, reg $\Longleftrightarrow$ normal.

Motivation. Polar coordinates on $\mathbb{R}^2$: $x = r\cos\varphi$, $y = r\sin\varphi$

"Classical" domain: $\underset{r}{\mathbb{R}_{\geq 0}} \times \underset{\varphi}{\mathbb{R}/2\pi\mathbb{Z}} \longrightarrow \mathbb{R}^2$ Problem: there are boundaries.

Modified domain: $(\mathbb{R}\times\mathbb{R})/\sim \xrightarrow{\pi} \mathbb{R}^2$ $(r,\varphi) \sim (-r, \varphi+\pi)$

$(r,\varphi) \longmapsto (r\cos\varphi, r\sin\varphi)$



$\pi|_{\{r\neq 0\}}$ iso onto $\mathbb{R}\setminus\{0\}$

$\pi^{-1}(0) = \{r=0\} = \mathbb{R}/\varphi\sim\varphi+\pi =$ "angles of the line through origin up to π" $= \mathbb{RP}^1$ as a topological space

Ex¹. $\pi^{-1}(C) = \tilde{C} \cup \{r=0\}$

In coordinates: $y^2 = x^3 + x^2$ apply $\pi^\#$:

$r^2\sin^2\varphi = r^3\cos^3\varphi + r^2\cos^2\varphi$ equation for $\tilde{C}$

$$\Leftrightarrow \quad 0 = \underbrace{r^2}_{\{r=0\}} \cdot \underbrace{\left(r\cos^3\varphi + \cos^2\varphi - \sin^2\varphi\right)}_{\tilde{C} \,\to\, \text{nonsingular}}$$

Ex². $C_2 = \{y^2 = x^3\} \xrightarrow{\pi^\#} r^2\sin^2\varphi = r^3\cos^3\varphi \Leftrightarrow r^2(r\cos^3\varphi - \sin^2\varphi) = 0$

$\tilde{C}_2 = \left\{ r = \dfrac{\sin^2\varphi}{\cos^3\varphi} \right\}$ $\dfrac{\sin^2\varphi}{\cos^3\varphi} \approx \dfrac{\varphi^2}{1}$ at the origin $\longrightarrow$ behaves like a parabola

$\longrightarrow \tilde{C}_2$ nonsingular

Algebraic Geometry I, lecture 28

Blow-up

Def. $\mathrm{Bl}_0 \mathbb{A}^2 := V(xv - yu) \subseteq \underset{x,y}{\mathbb{A}^2} \times \underset{u,v}{\mathbb{P}^1}$.

The projection $\pi : \mathrm{Bl}_0 \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ is called the blowup of $\mathbb{A}^2$ at O.

Rmk. Projection to the 2nd factor is iso to $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow \mathbb{P}^1$.

Rmk. $\pi^{-1}(0) \simeq \mathbb{P}^1 = E$ exceptional divisor

Lemma. $\pi|_{\pi^{-1}(A^2 \setminus \{0\})}$ is an iso onto $A^2 \setminus \{0\}$, with inverse (id, p) (see below).

In particular, $\mathrm{Bl}_0 \mathbb{A}^2$ is the closure of the graph of p.

$$p : A^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$$
$$(x,y) \longmapsto [x,y]$$

As morph of schemes: let $f \in k[x,y]$ be homog. Then

$p|_{D_{\mathbb{A}^2}(f)} : D_{\mathbb{A}^2}(f) = \mathrm{Spec}(A_f) \longrightarrow D_{\mathbb{P}^1}(f) - \mathrm{Spec}(A_f)_0$ is given by the inclusion map $(A_f)_0 \hookrightarrow A_f$.

Pf: $\pi^{-1}(D(x)) = V(xv - yu) \subseteq D(x) \times \mathbb{P}^1$

$$\Updownarrow$$

$$v = \frac{y}{x} u \Rightarrow u \neq 0 \Rightarrow \frac{v}{u} = \frac{y}{x}$$

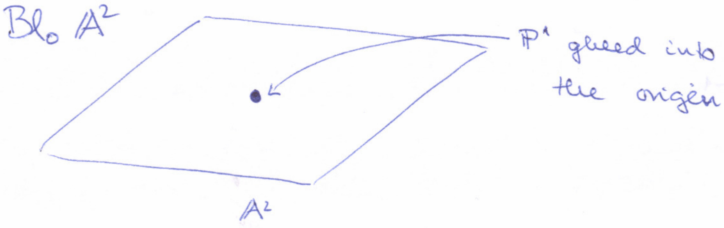
$$= V\left(\frac{v}{u} = \frac{y}{x}\right) \subseteq D(x) \times D(u)$$

$$= \text{graph of } D(x) = \mathrm{Spec}\, A_x \longrightarrow D(u) = \mathrm{Spec}\, \underbrace{(k[u,v]_u)_0}_{k[\frac{v}{u}]}$$

$$\frac{y}{x} \longleftarrow \frac{v}{u}$$

$$= \text{graph of } p|_{D(x)} \qquad \Rightarrow \pi^{-1}(A^2 \setminus \{0\}) = \text{graph of } p.$$

$\mathrm{Bl}_0 \mathbb{A}^2$



See picture in Hartshorne.

Let $C \subseteq \mathbb{A}^2$ be a curve, and define $\tilde{C} := \overline{\pi^{-1}(C \setminus \{0\})}$, called the proper transform of C.

Ex. 1. $y^2 = x^3 + x^2$.

On D(u): $V := \frac{v}{u}$ $\rightarrow$ $xv = yu$ yields $y = x\frac{v}{u} = xV$

$\pi^{-1}(C) \cap D(u)$ has equation $x^2V^2 = x^3 + x^2$

$$\iff \underbrace{x^2}_{E \cap D(u)} \underbrace{(V^2 - x - 1)}_{\tilde{C} \cap D(u)} = 0 \qquad \Rightarrow \tilde{C} \cap D(u) = V(V^2 - x - 1) \text{ nonsingular}$$

On D(x): $U := \frac{u}{v}$ $\Rightarrow$ $x = Uy$

$\pi^{-1}(C) \cap D(x)$: $y^2 = y^3U^3 + y^2U^2 \iff y^2 \cdot \underbrace{(yU^3 + U^2 - 1)} = 0$

$\tilde{C} \cap D(x)$ is nonsingular

$\boxed{\tilde{C} \text{ nonsingular}}$

Ex. 2. $y^2 = x^3$

On D(u): $x^2V^2 = x^3 \iff x^2 \cdot \underbrace{(x - V^2)} = 0$

parabola $\tilde{C}_2$, nonsingular

On D(x): $y^2 = U^3y^3 \iff y^2\underbrace{(U^3y - 1)} = 0$

nonsingular

$\Rightarrow \boxed{\tilde{C}_2 \text{ nonsingular}}$

Higher dimensions

The blowup of $\mathbb{A}^n$ at 0 is

$$\begin{tikzcd} Bl_0(\mathbb{A}^n) = V(x_iy_j - x_jy_i) \subseteq \overset{x_1, \dots, x_n}{\mathbb{A}^n} \times \overset{y_1, \dots, y_n}{\mathbb{P}^{n-1}} \arrow[d] \\ \mathbb{A}^n \end{tikzcd}$$

$\pi^{-1}(0) = \mathbb{P}^{n-1} = E$

$\pi^{-1}(\mathbb{A}^n \setminus \{0\}) =$ graph of $p: \mathbb{A}^n \setminus \{0\} \longrightarrow \mathbb{P}^{n-1}$

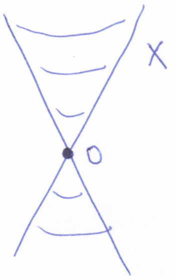
Local charts: $D(y_i) = \operatorname{Spec} k\left[x_i, \frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}\right]$

$x_iy_j = x_jy_i \Rightarrow x_i \cdot \frac{y_j}{y_i} = x_j$

Let $X = V(I) \subseteq \mathbb{A}^n$ closed reduced subscheme.

$\tilde{X} = \overline{\pi^{-1}(X \setminus \{0\})}$

Ex. 3. $X = \operatorname{Spec} k[x, y, z] / \underbrace{(x^2 + y^2 + z^2)}_{f}$ $\qquad \partial f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$



Algebraic Geometry 1, lecture 28

Let u, v, w be homog coordinates on $\mathbb{P}^2$.

On $D(u)$: $V := \frac{v}{u}$, $W := \frac{w}{u}$ $\quad xw = zu \Rightarrow z = xW$

$xv = yu \Rightarrow y = xV$

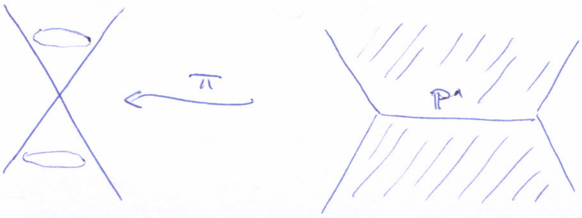
$\pi^{-1}(X) \cap D(u)$: $x^2 + x^2V^2 + x^2W^2 = 0$

$$\Leftrightarrow \underbrace{x^2}_{D(u)\cap E} \cdot \underbrace{(1+V^2+W^2)}_{\tilde{X} \cap D(u)} = 0$$

nonsingular hypersurface in $\mathbb{A}^3_{x,V,W}$

By symmetry, $\tilde{X} \cap D(v)$ and $\tilde{X} \cap D(w)$ are also nonsingular. $\Rightarrow \tilde{X}$ is nonsingular.

$\tilde{X} \cap E = \underbrace{V(u^2+v^2+w^2)}_{\mathbb{P}^1 \text{ (circle)}} \subseteq \mathbb{P}^2_{u,v,w}$



What is the normal bundle of $\tilde{X} \cap E =: E_X$ inside $\tilde{X}$? $N_{E_X} = T_{\tilde{X}}|_{E_X} / T_{E_X}$

Du Val singularities

G finite group, $G \curvearrowright X$, $X = \operatorname{Spec} A$ (equivalently, $G \curvearrowright A$)

The quotient of X by G is $X/G := \operatorname{Spec}(A^G)$ where $A^G = \{a \in A \mid a^g = a \ \forall g \in G\}$

(Appreciate how easy this definition is; in diff. geo, defining quotients is much harder.)

$k := \mathbb{C}$, $G \leq SL_2(\mathbb{C})$ finite subgp, $X := \mathbb{A}^2/G$, $Y :=$ minimal resolution of singularities of X obtained by repeatedly blowing up singular points, $\pi: Y \longrightarrow X$.

Hironaka: this process eventually terminates, $\omega_Y \simeq \mathcal{O}_Y$.

Claim 1. $\pi^{-1}(0) = C_1 \cup \dots \cup C_n$, $\forall C_i \simeq \mathbb{P}^1$ (curves)

Claim 2. (McKay correspondence)

Dual graph of $\pi^{-1}(0)$ is isomorphic to the McKay graph of G

Vertices $C_1, \dots, C_n$

Edges b/w $C_i, C_j \Leftrightarrow C_i \cap C_j \neq \emptyset$

Vertices $\rho_1, \dots, \rho_n$ nontrivi irred reps of G

Edges b/w $\rho_i, \rho_j \Leftrightarrow \rho_i \subseteq \rho_j \otimes \underset{\text{std. rep.}}{\mathbb{C}^2}$

Ex 4. (= Ex. 3) $G = \langle \underbrace{\begin{pmatrix}-1 & \\ & -1\end{pmatrix}}_{\xi} \rangle = \mathbb{Z}/2$

$\xi(x,y) = (-x,-y)$ $\mathbb{C}[x,y]^G = \mathbb{C}[\underset{z_0}{x^2}, \underset{z_1}{xy}, \underset{z_2}{y^2}] = \mathbb{C}[z_0,z_1,z_2]/_{z_1^2 = z_0 z_2}$

$\downarrow$

Ex. 3 up to change of variables

Dual graph

$X_E = \mathbb{P}^1$

$\longleftrightarrow$

McKay graph

$\mathbb{C}$ with action $g: \xi e = -e$